ENGN/PHYS 207 (November 30, 2022)
Frequency Sensitive Circuits: Phasor Math, Impedance, and RC filters

1 Wave In, Wave Out

So much of circuit analysis is predicated on putting sine waves into a circuit and seeing what wave comes out the other end. As we sweep over frequency measuring input and output waves, we characterize the frequency response of the circuit. So let's start with a simple picture illustrated in Figure 1. Clearly there are 2 sine waves here. Keep it in mind as we make our way through some hefty maths today.

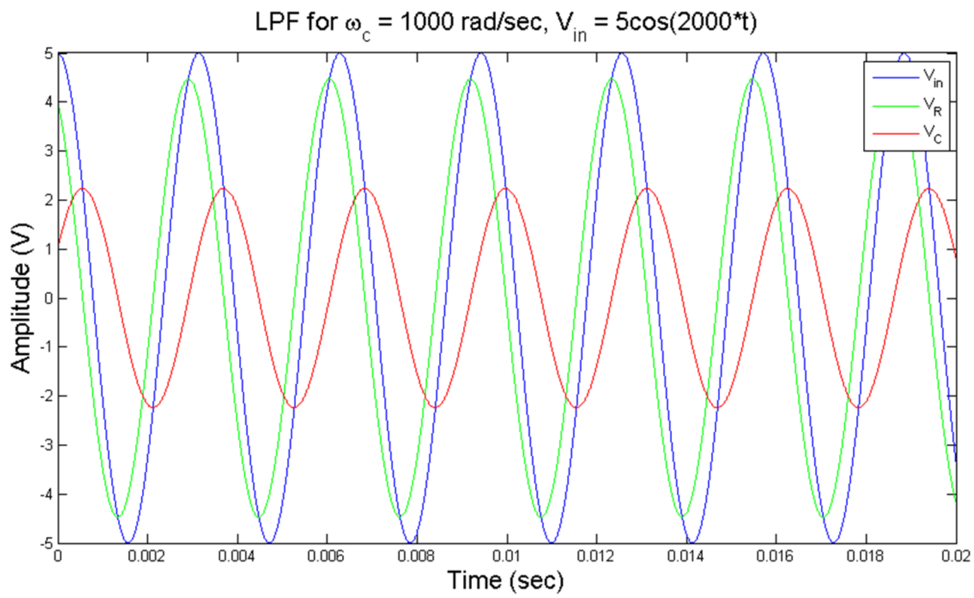

Figure 1: A familar looking scene from circuits lab. Blue trace: input signal. Red trace: output signal, in this case measured across the capacitor in an LPF set for $\omega_c = 1000$ rad/s or $f_c \approx 159$ Hz. Green trace: voltage across the resistor (we don't normally measurere or care too much about this signal since it is NOT the output of the LPF circuit.

1. List the 3 parameters that define a (co)sine wave? If you need a bigger hint, take a look at the equations below:

$$v_{in}(t) = r_{in}\cos(\omega t + \phi_{in}) \tag{1}$$
$$v_{out}(t) = r_{out}\cos(\omega t + \phi_{out}) \tag{2}$$

   Here, r is the amplitude of the wave, $\omega = 2\pi f$ is the angular frequency (rad/s) and ϕ is the phase angle or the initial phase of the wave (to see this just set $t = 0$).

2. Next question: Which of these 3 parameters differ between the input and output waves? Which of these parameter(s) are the same? Hopefully, you will see there are 2 parameters that have different values and 1 that has the same value.

By the way, we've cast the above discussion on waves in terms of input and output waves of a circuit (e.g. LPF), but these very well could have been the voltage and current across a capacitor, $v_c t$ and $i_c(t)$. In the first case, the transfer function $\tilde{H}$ encodes the relation between the voltage input and output waves. In the second case, it is the mysterious impedance $\tilde{Z}$ the encodes relation between voltage and current. As we've seen and experienced in lab, both are functions of (angular) frequency ω.

2 Math Stuff: What the heck is a phasor and what does it have to do with impedance and transfer functions?!

In the previous section, we concluded that input and output waves maintain the same frequency, but have different magnitude (amplitude) and phase. In other words the magnitude and phase of the input wave are altered as the wave is transferred through the circuit. *A phasor is really just a complex number the encodes both magnitude and phase represented in just a single complex number.*

With that prelude, it is time to dive into some math weeds. Don't worry no snakes, crocodiles, or other nasties hiding in here. In fact, we find one very pretty math picture, illustrated in Figure 2. Here we see an elegant 3-way equivalence between 1) drawing a triangle; 2) writing a complex number $z = a + jb$, and writing a complex exponential $re^{j\phi}$.

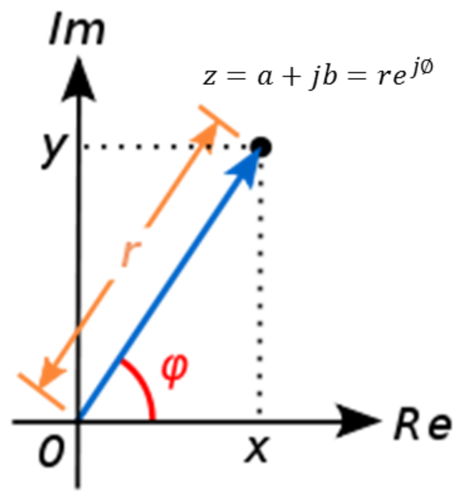

Figure 2: Complex number equivalent representations. Remember, it's all trinangles in the end!

1. Write an expression for the length of the adjacent and opposite triangle sides, a and b in terms of r and ϕ.
2. Next, flip the problem on its head: Write an expression for r and ϕ in terms of a and b. Note what we are really doing here is making use of Euler's formula!
3. Notice the $z = a + jb$ form that naturally occurs in circuits land. For example what is the equivalent impedance of a resistor R and capacitor C in series? (This is the backbone of the RC filters we studied and played with in lab!). Plot this point on the complex plane for $R = 1.5k\Omega$, $C = 68$ nF, and $\omega = 10^4$ rad/s.

4. Next, write this series impedance $\tilde{Z}_s$ in complex exponential form—basically solve for r and ϕ. Then draw it on the complex plane. Does it point to the same spot on the complex plane that you marked in the previous step? Spoiler alert: it should!

5. How would your answers above change if we set the angular frequency to $\omega = 100$ rad/s?

Before venturing on, watch what happens when we let the phasor rotate around at rate ω. This nice animation (on Wikipedia linked here) is a good visual to keep in mind as well! A static image is shown below (Figure 3). Phasor analysis is really just looking at a single snapshot in time—a static picture that details the amplitude and phase of a wave. Again, we understand it rotates around in time at rate ω to actually carve out a sine wave in time. Here's the take home message: $re^{j\phi}$ is a **phasor**. It is a natural representation of magnitude and phase. Remember, these are exactly the parameters we need to keep track of when we quantify the relation of two waves, per the previous section.

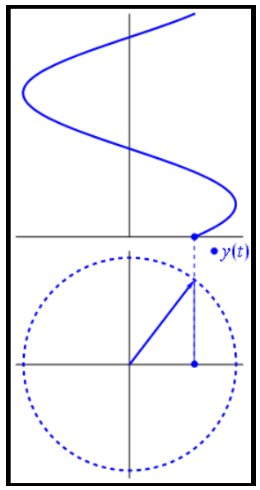

Figure 3: A phasor rotates in time at rate ω carving out a (co)sine wave in time. Image credit: `https://en.wikipedia.org/wiki/Phasor`

3 A little deeper math dive into complex exponentials

Complex exponentials are ultimately just a very convenient mathematical tool for expressing sines and cosines. Again, just look back at Figure 3 and the animation thereof! That's where this game all started. Recall we need to know 3 things to define a wave: 1) amplitude; 2) phase; 3) (angular) frequency. Using Euler's formula $e^{j\omega t} = \cos\omega t + j\sin\omega t$ we can write oscillating sine wave signal as follows:

$$r\cos(\omega t + \phi) = Re[re^{j(\omega t+\phi)}] = Re[\underbrace{re^{j\phi}}_{phasor}\, e^{j\omega t}] \tag{3}$$

where r is the magnitude, $e^{j\phi}$ ecodes the phase angle, and $e^{\omega t}$ expresses the time dependence, thus creating an actual wave. Notice the ***time dependence can be written separately from the magnitude and phase information***. This is the mathemagic behind phasors and—believe it or not—is the sole reason they are so pervasive in physics and engineering!

Exponentials also have some very nice properties for multiplying and dividing complex numbers:

$$z_1 z_2 = \left(r_1 e^{j\phi_1}\right)\left(r_2 e^{j\phi_2}\right) = r_1 r_2 e^{j(\phi_1+\phi_2)} \tag{4}$$

Note the amplitudes multiply; the angles add.

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\phi_1}}{r_2 e^{j\phi_2}} = \frac{r_1}{r_2} e^{j(\phi_2-\phi_2)} \tag{5}$$

Note the amplitudes divide; the angles subtract.

Got all that? :) Now, sketch each of the following in the complex plane. It may be helpful to write in $z = a + jb$ in complex exponential form $re^{j\phi}$. **Remember, r represents the magnitude and $e^{j\phi}$ represents the phase angle (rotation) relative to positive real axis**!

1. $z_1 = 5\, e^{-j\pi/2}$
2. $z_2 = 1 + 1j$
3. $z_3 = \frac{1}{1+j}$

You may note that z_1 above looks supiciously like $1/j\omega C$ with values plugged in; and z_3 bears quite the resemblence to the transfer function of a low pass filter. Hmmm, *muy interesante*!

4 Transfer function: Low Pass Filter example

We've done a lot of math preliminaries. Now let's try to answer our core question: What the heck is a transfer function anyway?! We'll use the venerable single stage passive LPF as an example to gain some general insight. We've previously seen that the transfer function is given by:

$$\tilde{H}_{lpf}(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega/\omega_c} = \frac{1}{1 + jf/f_c} \tag{6}$$

Let's also recall another important definition, that of a transfer function. In the context of the LPF, this is just the relation between the input and output waves as follows:

$$\tilde{H}_{lpf} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$$

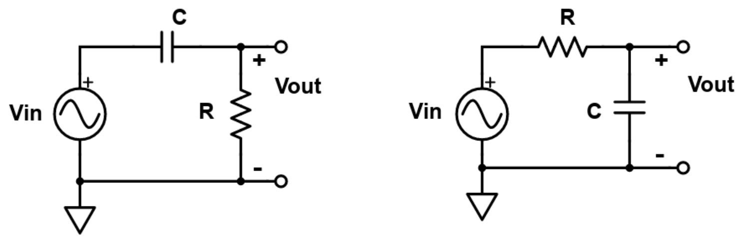

Figure 4: Venerable single stage passive RC filters. By this point, you shoudl be able to readily identify which of these is high and low pass, respectively.

Perhaps more elucidating is to write this in the form:

$$\tilde{V}_{out} = \tilde{H}_{lpf}\tilde{V}_{in} \tag{7}$$

It is also super important to recall that we're really talking about two waves that oscillate in time per Eqn 1, repeated here for convenience with the added complex exponential notation:

$$v_{in}(t) = r_{in}\cos(\omega t + \phi_{in}) = Re[\underbrace{r_{in}\, e^{j\phi_{in}}}_{\tilde{V}_{in}}\ e^{j\omega t}]$$

$$v_{out}(t) = r_{out}\cos(\omega t + \phi_{out}) = Re[\underbrace{r_{out}\, e^{j\phi_{out}}}_{\tilde{V}_{out}}\ e^{j\omega t}]$$

Note the phasors $\tilde{V}_{in}$ and $\tilde{V}_{out}$ are keeping track of the amplitude and phase of the input and output waves, as denoted above. Now, here is the magic, the transer function encodes the relation between the input and output at a certain frequency $\omega = 2\pi f$. That's what Eqn 7 actually says in very compact mathematical terms. Amazing!

Now let's make this a little less abstract by working through an example. Assume we have an LPF set for a cutoff frequency of $f_c \approx 159$ Hz or $\omega_c = 1000$ rad/s. Let's say you are at the lab bench testing your LPF to characterize its behavior. Currently you have the frequency of the wave generator set to f =10 Hz and an amplitude of r_{in} = 100 mV. Per convention, we'll assume the phase angle is given by $\phi_{in} = 0$.

1. Firstly, before getting into computations, given the frequency of the input wave, as well as the filter cutoff, would you expect this signal to pass or be significantly attenuated?
2. Compute $\tilde{H}_{lpf}$ for this scenario and plot the result in the complex plane. It may be helpful here to use the nice properties dividing complex exponentials. (See Eqn 6)
3. On another axis, plot the phasor representing the input wave.
4. Next compute the magnitude and phase of the output wave (using Eqn 7). Plot the phasor on the same axis as the one with the input wave. Be careful to note its magnitude and phase. It may be helpful here to use the nice properties of multiplying complex exponentials to make this process relatively painless. Plot the resulting phasor $\tilde{V}_{out}$ on the same axis.

5. Using your results above, make a quick sketch of the input and output waves vs time, similar to what is shown in Figure 1, and what are you now so accustomed to seeing on the picoscope display in lab. One important question: what is the period of this wave. Remember $T = 1/f$. Be sure to indicate the proper time scale in your sketch. Basically, you are just trying to make a quick visual showing the relation between magnitude and phase of the input and output waves.

6. Let's interpret the result of all this math madness. How does your purely math result stack up against your response for the first question in this section? Does it match your experience in the lab?

7. Now repeat steps 1-6, but this time imagine you have increased the wave generator frequency to $f = 1590$ Hz.

Hopefully, this little exercise has illuminated the math and theory behind transfer functions!

5 Phase Shift

In the above example, and through lots of experience in lab, you know that the output wave is phase shifted relative to the input. But what's the math behind it? Can we predict how much the phase will shift as a function of frequency? Of course! You already handled 2 special cases above, but let's generalize. Again, we'll use the LPF as a motivating example. The same exact concepts can be extended to any filter.

Recall we wrote the magnitude and phase relation between input and output as:

$$\tilde{V}_{out} = \tilde{H}_{lpf}\tilde{V}_{in} \tag{8}$$
$$r_{out}e^{j\phi_{out}} = (r_H e^{j\phi_H})(r_{in}e^{j\phi_{in}}) \tag{9}$$
$$= \underbrace{(r_H r_{in})}_{ampl.}\ \underbrace{e^{j(\phi_H+\phi_{in})}}_{phase} \tag{10}$$

The last 2 equatinos just explicitly cast everything in complex exponential form (eyes glazed over yet?!). Here, we are primarily intersted in analyzing the phase. Again, by convention, we assume $\phi_{in} = 0$. This implies that the phase shift is simply given by phi_H. Ok, great, how do we solve for that? Well, we just have to go back to our circuits roots and recall that for the LPF:

$$\tilde{H}_{lpf}(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega/\omega_c} = \frac{1}{1 + jf/f_c} \tag{11}$$

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We could attepmt to put the transfer function into $z = a + jb$ form multiplying top and bottom by complex conjugates. This works, but it gets messy quickly. Furthermore, imagine trying that trick with a more complicated looking transfer function, such as a BPF. More like BARF.

So let's recast in complex exponential form:

$$\tilde{H}_{lpf}(\omega) = \frac{1}{1 + j\omega/\omega_c} \quad (12)$$

$$r_H\, e^{j\phi_H} = \frac{a_N\, e^{j\phi_N}}{a_D\, e^{j\phi_D}} \quad (13)$$

$$= \frac{a_N}{a_D} e^{j(\phi_N - \phi_D)} \quad (14)$$

where N and D denote numerator and denominator. The approach casting everything as a complex exponential breaks down a bigger problem into a series of smaller, easier ones. Watch this!

1. Compute the phase angle of the numerator ϕ_N. Draw the phasor in the complex plane, if it helps! (And don't overthink this!)
2. Do the same for the denominator. Again, draw the phasor and analyze a triangle to make this really simple—it's just basic trig, I promise.
3. Now write an expression for the transfer function phase angle ϕ_H. This is the phase shift, which is a function of frequency! Need a hint, just look up at the previous set of equations, namely the bottom-most.
4. Make a quick sketch the phase shift ϕ_H as a function of frequency.
5. Gut check - does this make any sense? What is the phase shift for when the wave generator is set to 10 Hz? 1590 Hz? How does this pure math result square with your experience in lab (perhaps staring at picoscope displays for too many hours)?
6. Why does all of this even matter, you ask? Here's one good motivating example: Imagine you are processing human speech (massive applications for this, needless to say). Human speech is composed of a superposition of many waves, not just a single sine wave. Each of these components will phase shift a bit differently according to its frequency. So when you run speech through a filter, it comes out the other end a bit distorted. In this case, distortion is not good (we're not trying to make a Jimmy Hendrix vintage guitar amp here!)—it will muddle the sound quality and possible human or computer ability to understand and process the spoken word. Can you think of another real life example where a packet of wideband waves is the input to a system and you don't want to see distortion? Hint, there's one inside your body, ticking away about once per second. For a more in depth look, check out the ECG example on this page. Imagine what happens if all these component waves start phase shifting relative to each other. What if your cardiologist saw a distorted QRS complex? Is it your heart, or just an effect of the filter? Hopefully, the good folks who designed the ECG circuitry did a good job to minimize phase distortion in the frequency band that matters—this may be a literal life and death design decision.

OK, that was a sort of heavy end. Really, though, math and circuits is powerful, potentially life-saving stuff. That's why we study it, right? This concludes our long and windy but hopefully elucidating tour through phasors, transfer functions and phase shifts. Thanks for coming along for the ride and come back anytime!