ENGN/PHYS 207 (November 30, 2022)
SOLUTION: Frequency Sensitive Circuits: Phasor Math, Impedance, and RC filters

1 Wave In, Wave Out

1. List the 3 parameters that define a (co)sine wave? If you need a bigger hint, take a look at the equations below:

   1. amplitude (r), 2. frequency ($\omega = 2\pi f$), 3. phase (ϕ)

2. Next question: Which of these 3 parameters differ between the input and output waves? Which of these parameter(s) are the same?

   Frequency of input and output is the SAME! The other 2 can be different. Hopefully, you this gibes with intuition based on building and measuring in lab!

2 Math Stuff: What the heck is a phasor and what does it have to do with impedance and transfer functions?!

1. Write an expression for the length of the adjacent and opposite triangle sides, a and b in terms of r and ϕ.

$$a = r\cos\phi$$

$$b = r\sin\phi$$

   It really is basic trig. See, complex numbers aren't that bad :)

2. Next, flip the problem on its head: Write an expression for r and ϕ in terms of a and b.

$$r = \sqrt{a^2 + b^2}$$

$$\phi = \tan^{-1}(b/a)$$

   Still just basic trig, fun and easy, no?

3. Notice the $z = a + jb$ form that naturally occurs in circuits land. For example what is the equivalent impedance of a resistor R and capacitor C in series? (This is the backbone of the RC filters we studied and played with in lab!). Plot this point on the complex plane for $R = 1.5k\Omega$, $C = 68$ nF, and $\omega = 10^4$ rad/s.

$$\tilde{Z}_s = \tilde{Z}_R + \tilde{Z}_C = R + \frac{1}{j\omega C} = R - \frac{j}{\omega C}$$

   Last step just multiplies top and bottom by j. Notice the imaginary part is frequency dependent! That's the whole concept of impedance.

$$\frac{1}{\omega C} = \frac{1}{68 \times 10^{-5}\ \text{Frad/s}} \approx 1470\ \Omega$$

   Note the unit is Ohms. You can convince yourself of this by working through units.

Anyway, to draw this we would go over 1.5kΩ on the real axis, then down the imaginary axis about the same amount (because it's minus j). This point sits almost due SE in the complex plane. and each side of the triangle is about 1.5kΩ.

4. Next, write this series impedance $\tilde{Z}_s$ in complex exponential form—basically solve for r and ϕ. Then draw it on the complex plane. Does it point to the same spot on the complex plane that you marked in the previous step? Spoiler alert: it should!

$$r = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \approx 2100\ \Omega$$

   (Note the j does NOT appear in the above calculation). It never should. The j only denotes which part is the imaginary part. Everything that hits the j is the actual magnitude along the imaginary axis.

5. How would your answers above change if we set the angular frequency to $\omega = 100$ rad/s? The imaginary part changes; the real part doesn't. Let's recompute. We know ω decreased by a factor of 10, so that $1/\omega C \approx 147000\ \Omega$ increased by a factor of 10. (This means the capactor's impedance increased at a lower frequency; makes sense)

   The imaginary part is now 100 times bigger, and now also about 100 times larger than the real part $R = 1500\ \Omega$ So the phasor points nearly along the negative imaginary axis and has magnitude of $\approx$ 147000 (in units of kiloohms). It's actually a little more and you could use Pythagorean formula to formally compute. But you'll find only a small contribution from the real part.

3 A little deeper math dive into complex exponentials

1. $z_1 = 5\, e^{-j\pi/2}$

   Can just read off magnitude of 5 and phase shift of $-\pi/2$. So this points 5 nits along the negative imaginary axis (due South).

2. $z_2 = 1 + 1j$

   Read this out loud, it says "over 1 on the real axis, up one on the imaginary." This points due NE. Formally, phase angle is $\pi/4$ with a magnitude $\sqrt{2}$. We can also write this as $z_2 = \sqrt{2}e^{j\pi/4}$

3. $z_3 = \frac{1}{1+j}$

   This one is a little trickier due to the division. You could use complex conjugates, but that's a pain . Intead just realize the denominator is the same as z_2 above. Also note that $1 = 1e^{j0}$ as it just lies on the real axis. Then just divide numerator and denominator, super easy to do using properties of exponentials. We quickly see $z_3 = \frac{1}{\sqrt{2}}e^{j(-\pi/4)}$. This is a phasor that points due SE and has a magnitude of $1/\sqrt{2}$.

You may note that z_1 above looks supiciously like $1/j\omega C$ with values plugged in; and z_3 bears quite the resemblence to the transfer function of a low pass filter. Hmmm, *muy interesante*!

4 Transfer function: Low Pass Filter example

$$\tilde{H}_{lpf}(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega/\omega_c} = \frac{1}{1 + jf/f_c} \tag{1}$$

Assume we have an LPF set for a cutoff frequency of $f_c \approx 159$ Hz or $\omega_c = 1000$ rad/s. Let's say you are at the lab bench testing your LPF to characterize its behavior. Currently you have the frequency of the wave generator set to f =10 Hz and an amplitude of $r_{in} = 100$ mV. Per convention, we'll assume the phase angle is given by $\phi_{in} = 0$.

1. Firstly, before getting into computations, given the frequency of the input wave, as well as the filter cutoff, would you expect this signal to pass or be significantly attenuated?

   A frequency of 10 Hz is well within the passband of a LPF. So we'd expect the signal to pass: the output wave amplitude should be nearly the same as the input. We also expect the phase shift to be minimal.

2. Compute $\tilde{H}_{lpf}$ for this scenario and plot the result in the complex plane. It may be helpful here to use the nice properties dividing complex exponentials. (See Eqn 1)

   First compute the ratio $f/f_c = 10/159 \approx 0.063$. Then plug into: $\tilde{H}_{lpf}(\omega) = \frac{1}{1+jf/f_c} = \frac{1}{1+j0.063}$
   The denominator is pretty obviously close to 1, which implies the transfer function is pretty close to 1, which implies the input and output are nearly the same, as expected. But onward we go with calculations. Let's work the denominator. It's magnitude is:

   $$r_D = \sqrt{1^2 + (0.063)^2} \approx 1.002$$

   Phase angle is

   $$\phi_D = \tan^{-1}(0.063)$$

   This computes to be about 0.063 rad or about 3.6 degrees. Small phase shift!

   Put it together we can write this in phasor form as: $1.002e^{j\,0.063}$.

   The numerator in phasor form is just $1e^{j0}$. Now divide numerator and denominator and we get

   $$\tilde{H}_{lpf}(\omega = 1000) = \frac{1}{1.002}e^{j(0-0.063)} \approx 0.998e^{-j\,0.063}$$

3. Plot the phasor representing the input wave. This has nearly unit length, just a smidge less. It lies just below the real axis shifted about 3.6 degrees CW off the real axis.

4. Next compute the magnitude and phase of the output wave (using Eqn **??**). Be careful to note its magnitude and phase. It may be helpful here to use the nice properties of multiplying complex exponentials to make this process relatively painless. Plot the resulting phasor $\tilde{V}_{out}$ on the same axis.

   $$\tilde{V}_{out} = \tilde{H}_{lpf}\tilde{V}_{in} = (0.998e^{-j\,0.063})(100\,e^{j0}) = 99.8e^{-j\,0.063}$$

   This means the amplitude of the output wave is 99.8 mV, nearly the same as the 100 mV input (as expected!). The output is phase shifted just a few degrees. Note the negative phase shift means it LAGS in time...the input hits a peak and then shortly after the output does.

5. Using your results above, make a quick sketch of the input and output waves vs time, similar to what is shown in Figure 1, Basically you are drawing the waves:

$$v_{in}(t) = 100\cos(1000t)$$

$$v_{out}(t) = 99.8\cos(1000t - 0.063)$$

   Obviously, they will very nearly overlap. One important question: what is the period of this wave. Remember $T = 1/f$. Be sure to indicate the proper time scale in your sketch.

6. Let's interpret the result of all this math madness. How does your purely math result stack up against your response for the first question in this section? Does it match your experience in the lab?

   Matches everything we've seen. The math don't lie! That's a new hit song title, by the way (cf. Shakira)

7. Now repeat steps 1-6, but this time imagine you have increased the wave generator frequency to $f = 1590$ Hz.

The main thing is the realize the ratio $f/f_c = 10$. This makes the denominator of the transfer function equal to $1+j10$. It has a magnitude $\sqrt{101} \approx 10$. So the magnitude of the transfer function is now about 1/10. This makes the output wave about 1/10 that of the input or about 10 mV. Strong attenuation as expected. 1590 Hz is well into the cutoff region for a 159 Hz cutoff frequency LPF. Note the angle of the transfer function is getting close to $-\pi/2$ (actually just $-\tan -110$, but close enough). So the output is going to be phase shifted nearly 90 degrees. This makes sense, everytime we see wave getting cutoff, we note a phase shift that comes with it. Again, hopefully the math convinces you that the math describes what the circuit is actually doing.

5 Phase Shift

1. Compute the phase angle of the numerator ϕ_N. Draw the phasor in the complex plane, if it helps! (And don't overthink this!)

   We already did this in the previous section. It is $\phi_N = 0$.

2. Do the same for the denominator. Again, draw the phasor and analyze a triangle to make this really simple—it's just basic trig, I promise. We basically already did this in the previous section too. Denominator points 1 on the real axis, then goes up the imaginary axis an amount $f/f_c = \omega/\omega_c$. So the phase angle

$$\phi_D = \tan^{-1}(f/f_c)$$

3. Now write an expression for the transfer function phase angle ϕ_H. This is the phase shift, which is a function of frequency! Need a hint, just look up at the previous set of equations, namely the bottom-most.

$$\phi_H = \phi_N - \phi_D = 0 - \tan^{-1}(f/f_c) = \tan^{-1}(f/f_c)$$

4. Make a quick sketch the phase shift ϕ_H as a function of frequency. The phase shift plot starts at 0, then snakes down like an inverse tangent does toward $-\pi/2$. It passes through $-\pi/4$ at the cutoff frequency (where $f = f_c$).

5. Gut check - does this make any sense? What is the phase shift for when the wave generator is set to 10 Hz? 1590 Hz? How does this pure math result square with your experience in lab (perhaps staring at picoscope displays for too many hours)? Yep - check.

6. For you to explore on your own.