The Circuits Big Three: KCL, KVL, Ohm's Law

1. **Ohm's Law**

   Ohm's Law relates three fundamental circuits quantities: voltage, current, and impedance. It is absolutely critical to understand and be able to apply the conceptual framework illustrated in Figure 1.
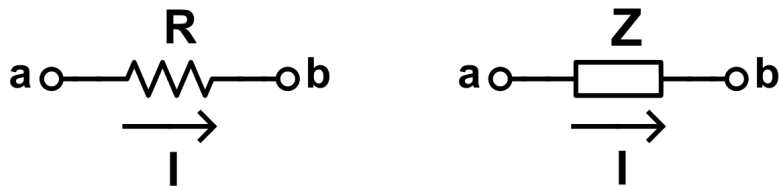

Figure 1: Ohm's Law Illustrated with resistor (left) and generalized impedance element (right).

   Conventional current (positive charges) flow down hill. So we write for a resistor:

$$V_R = V_a - V_b = IR \tag{1}$$

   For generalized impedance elements we similarly write:

$$\tilde{V}_Z = \tilde{V}_a - \tilde{V}_b = \tilde{I}\tilde{Z} \tag{2}$$

   Note we can also write phasors in complex exponential form to reveal ***magnitude*** and ***phase*** relations:

$$ve^{j\phi_v} = (ie^{j\phi_i})(ze^{j\phi_z}) = iz\ e^{j(\phi_i + \phi_z)} \tag{3}$$

   where v is the amplitude of the *voltage across* the impedance element, sinusoidally varying in time. Similarly, i reprsents the amplitude of the *current flow through* the impedance element, also sinusoidally varying in time. Finally, z represents the magnitude of the impedance, and $\phi_z = \phi_v - \phi_i$ is the phase shift between the current and voltage.

2. **KCL**

   Inspect any ***node*** (where two or more circuit elements connect). Total current flowing into the node must equal the total current flowing out of the node. This is just conservation of charge. For example, if i_1, i_2, i_3 are flowing in, and i_4 and i_5 are flowing out, then we can write either: $i_1 + i_2 + i_3 - i_4 - i_5 = 0$ or perhaps more intuitively, $i_1 + i_2 + i_3 = i_4 - i_5$.

3. **KVL**

   Inspect any ***complete loop***. Total energy gains must equal total energy losses. This is just a restatement of conservation of energy. Note that we do NOT need to transverse circuit

elements to complete a loop. We can also travel between two nodes that are not connected in the circuit to complete a loop in space.

4. **Parallel and Series equivalents** Series resistors mean there is more total resistance to go through, so these add to make a single series equivalent: $R_s = \sum_n R_n$.

   Parallel resistors means there are more paths for current flow, so the overall equivalent resistance is decreased. The parallel equivalent is found using: $1/R_p = \sum_n 1/R_n$.

5. **Series vs. Parallel Comparison**

Series

- Single path for current to flow through 2 or more elements from point a to b

- Voltage across elements can be different

- Current is the same

Parallel

- Elements share both nodes, multiple pathways for current to flow from point a to b

- Voltage is the same
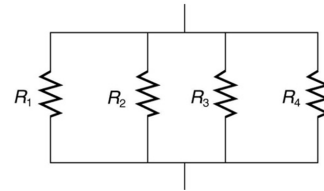
- Current can be different

Figure 2: Summary and Comparison of Series and Parallel Elements in Electrical Circuit

6. **Power Considerations**

   (a) Batteries are always rated for their two things: 1) their voltage (e.g. "9V" battery); and 2) their charge capacity: $\Delta Q = I\Delta t$. The latter is often specified in units of mA-hr. For example, a 2000 mAh battery can operate at 20 mA for 100 hours, or 200 mA for 10 hrs, until the battery is totally drained (no charge left).

   (b) Power dissipation occurs in resistors and LEDs. Resistors heat up. LEDs give off photons (light energy). Fundamentally, power is the amount of work done (energy transferred) per unit time. In circuits land, we write this as $P = IV$. Think waterfalls: More current dropping from a larger height produces more power. For electrical Circuits, We can use Ohm's Law $V = IR$ to get $P = IV = I^2R = V^2/R$. Be careful applying this formula to a single element at a time! The V refers to the voltage across a single element (not the power source!). The I refers to how much current flow through that specific element (not the total current flow in the circuit!)

DC Circuits You'll Find Everywhere

1. **Voltage Divider**

   The ubiquitous topology that exists everywhere in circuits. It is so often used because it transfers some fraction of input signal to the output, depending on the ratio of resistances.
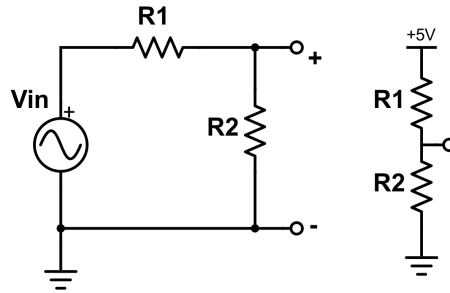

Figure 3: Voltage dividers drawn two different ways. Output V_{out} is measured across resistor R2 in both cases. The circuit on right shows a +5V source for V_{in}.

$$V_{out} = V_{R_2} = \left(\frac{R_2}{R_1 + R_2} \right) V_{in} \tag{4}$$

   Note, the voltage divider in Figure 3, could be generalized with two impedance elements $\tilde{Z}_1$ and $\tilde{Z}_2$.

2. **Wheatstone Bridge** The WB (Figure 4) is very commonly used topology to sense change in resistance. Resistance may be a surrogate for changing light intensity; humidity; material properties, mechanical deformation, etc. The main claim to fame of the WB is that its output is (approximately) linear with the change in resistance. Linearity is highly prized in many physics and engineering applications.
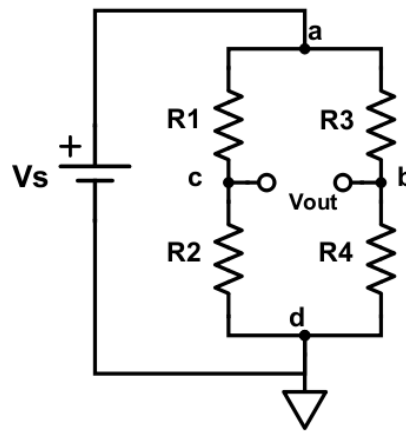

Figure 4: Wheatstone Bridge. Output is measured as the difference in voltage between nodes b and c.

Battery Ratings: voltage and charge capacity

Most batteries come with a rating for voltage, which is entirely familiar. How long will the battery last? Pick up a typical LiPo battery, for instance, and you'll see something like "500 mAh" printed on the side. Note the units are current (mA) × time (h).

In other words $Q_{battery} = I_{avg}\Delta t$ or rearrange to find the discharge time (life time) of the battery:

$$\Delta t = \frac{Q_{battery}}{I_{avg}}$$

Well, that depends how much current is flowing. If you have a system that flows an average current $I_{avg} = 50$ mA, then the battery will last for 10 hr (=500 mAh/50 mA). If your system flows 100 mA, the battery will drain in 5 hrs; if the current is a mere 100 μA = 0.1 mA, then your battery will last 5000 hr or about 208 days!
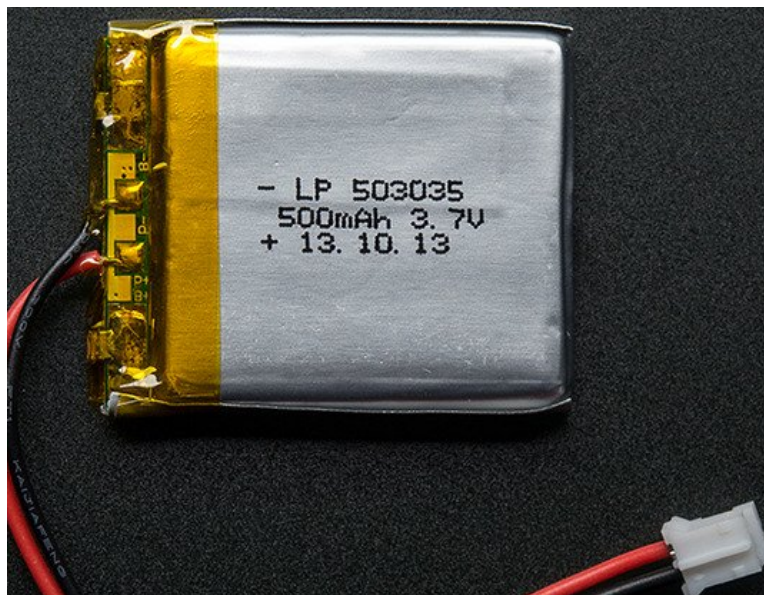

Figure 5: Single cell 3.7 LiPo battery. Note the 500 mAh charge capacity rating. Image credit:https://www.adafruit.com/product/1578.

Other fun and intersting things about seemingly ubiquitous LiPo batteries you should know:

1. The voltage of a fully charged LiPo battery is ≈ 4.2 V. This quickly falls to the nominal 3.7 V for most of the charging lifecycle.

2. Be careful with the polarity! The keyed JST connector (little white connector with a ridge) can differ between manufacturers. Reverse polarity can cause fires–yikes. More safety tips at this helpful Adafruit tutorial

Phasors, Impedance, Resonators and Filters

1. **Phasors and Complex Numbers**

   All you need to understand about phasor math is encoded in Figure 6. Note that complex exponential form (Euler's Identity) makes the math much easier thanks to nice properties such as $e^a e^b = e^{a+b}$, etc. Also remember to think about triangles! It also makes the math really easy.

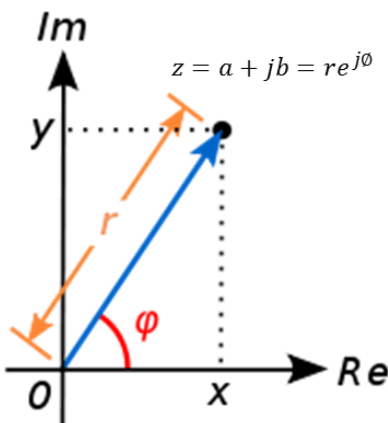   In general, phasors are used in circuits land to encode ***magnitude*** and ***phase***.

Figure 6: Complex Number/Phasors. Diagram shows three equivalent ways of writing/drawing z.

2. Impedance

   Impedance $\tilde{Z}$ is a sort of ***frequency dependent*** resistance. It just tells us how difficult it is to flow current through a certain element based: $\tilde{I} = \tilde{V}/\tilde{Z}$ The higher the impedance magnitude, the smaller the current. Impedance is a complex quantity because it encoded both magnitude and phase relations—the voltage and current can be out of phase with one another.

   Table 1 summarizes the impedance and limiting case behavior for R, C, and L.

	Resistor	Capacitor	Inductor
Symbol	R	C	L
Physical Operation	$v = iR$	$i = C(dV_c/dt)$	$v_L = L(di_L/dt)$
Impedance	$\tilde{Z}_R = R$	$\tilde{Z}_C = \frac{1}{j\omega C}$	$\tilde{Z}_L = j\omega L$
$|\tilde{Z}|$ $(\omega = 0)$	-	open circuit	short circuit
$|\tilde{Z}|$ $(\omega \to \infty)$	-	short circuit	open circuit

Table 1: Impedance Summary with limiting case behavior for very low $(\omega = 0)$ and very high $(\omega \to \infty)$ frequency signals.

3. **Series and Parallel Equivalents with Impedance** KVL, KCL, Ohm's Law work just like they did before, except now we have $\tilde{V} = \tilde{I}\tilde{Z}$. So all of the series and parallel equivalents are the same as before, except with Z's instead of R's.

   For series elements, we have: $\tilde{Z}_s = \tilde{Z}_1 + \tilde{Z}_2 + \ldots$.

   In parallel, we have: $1/\tilde{Z}_p = 1/\tilde{Z}_1 + 1/\tilde{Z}_2 + \ldots$.

Passive RC Filters

1. **RC filter flavors** Figure 7 shows 3 common passive RC filters: 1-stage low pass filter (LPF); 1-stage high pass filter (HPF); and band pass filter (BPF) constructed form cascaded LPF + HPF. In terms of filter design, two defining characteristics of any filter are 1) cut off frequency; 2) slope in attenuation region (how "hard" the filter cuts off).

Transfer Function $\tilde{H}$

Transfer functions represent the relation between input and output as a function of frequency: $\tilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$. The transfer function encodes the **magnitude** response $|\tilde{H}(\omega)|$, and **phase** response $\phi(\omega)$. You should feel very comfortable generating plots for each based on a given transfer function; you should also be able to look at a plot of decibel gain and instantly know what kind of filter is at hand.
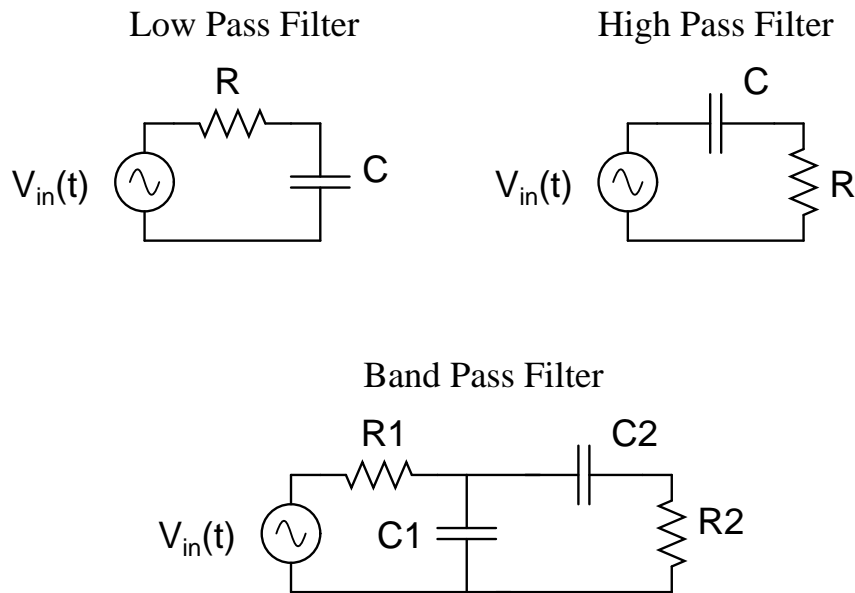
Low Pass Filter   High Pass Filter

Band Pass Filter

Figure 7: Basic passive RC filters. Output of LPF measured across C. Output of HPF measured across R. Output of BPF measured across R_2.

(a) **LPF**:
   $\tilde{H}_{LPF}(\omega) = \frac{1}{1+j\omega RC}$
   Cutoff frequency: $f_{lpf} = \frac{1}{2\pi RC}$.
   Slope in roll-off (attenuation) region $= 20$ dB/dec.

(b) **HPF**:
   $\tilde{H}_{HPF}(\omega) = \frac{j\omega RC}{1+j\omega RC}$
   Cutoff frequency: $f_{hpf} = \frac{1}{2\pi RC}$.
   Slope in roll-off (attenuation) region $= 20$ dB/dec.

6

(c) **BPF**:

$$\tilde{H}_{BPF}(\omega) = \left(\frac{1}{1+j\omega R_1 C_1}\right)\left(\frac{j\omega R_2 C_2}{1+j\omega R_2 C_2}\right)$$

Band pass frequencies: $f_L = f_{hpf} = \frac{1}{2\pi R_2 C_2}$; $f_H = f_{lpf} = \frac{1}{2\pi R_1 C_1}$

Slope in roll-off (attenuation) regions $= 20$ dB/dec on either side.

2. **Decibel Gain**

$$G(\omega) = 20 \log_{10}\left|\tilde{H}(\omega)\right| = 20 \log_{10}\left|\frac{\tilde{V}_{out}}{\tilde{V}_{in}}\right| \tag{5}$$

Decibel gain just works on a logarithmic scale to describe the amplitude vs. frequency behavior of the circuit. It just gives each order of magnitude equal footing. For instance going from to $|\tilde{H}| = 1$ to 10 takes up as much space on a graph as going for 10 to 100. Ditto from going from 1 to 1/10 or 1/10 to 1/100. Units are in decibels (dB).

3. **10x rule; cascades** Higher order passive RC filters can be constructed from single stages. For instance, we can make a 2-stage HPF by cascading two single RC-stage. The 2-stage cuts off twice as hard as the 1-stage (40 dB/dec slope). A three stage would cutoff at 60 dB/dec.

Proper cascades of passive filters require we adhere to the $10\times$ rule of thumb: each successive stage we add on requires the resistor to be (at least) a factor of $10\times$ greater than the one in the previous stage. This prevents current from leaking out of the earlier stages, which would torpedo its operation. Figure 8 shows one good example of a 3-stage HPF.
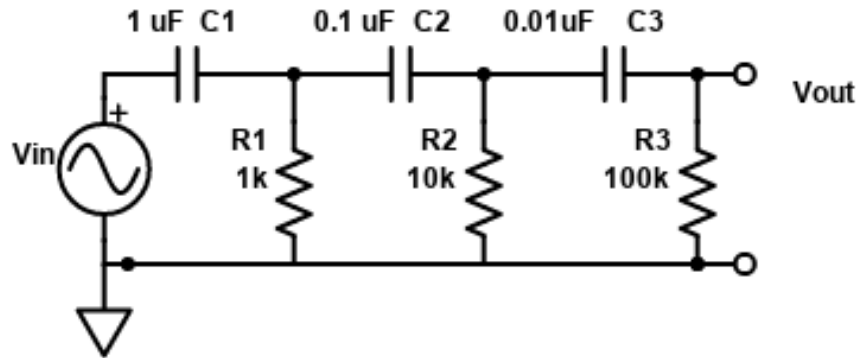

Figure 8: 3-stage aka 3rd order HPF. Note the resistors increase by $10\times$ in each successive stage, while the RC product remains the same.
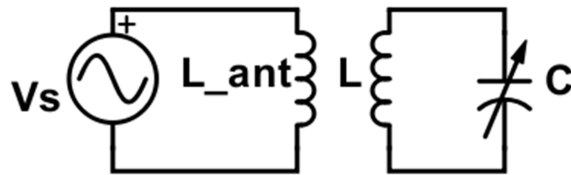
LC Resonator

Figure 9: Parallel LC resonator. Here, the capacitance is shown as variable, which reflects 2 real life applications: 1) turning a knob on a radio dial which sets the capacitance, hence the resonance frequency to tune into a particular radio station frequency; 2) variable resistance based on material properites of the dielectric insulator material, e.g. the "moonshine detector" circuit.

The classic resonator is formed by a parallel combo of inductor and capacitor. The inductor and capacitor trade energy stored in magnetic and electric fields. There is a rate at which these elements "want" to oscillate. This frequency is the one at which the $|\tilde{Z}_L| = |\tilde{Z}_C|$. This is the **resonance frequency**:

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

Resonators show large voltage response (output) only when driven at, or very near, the resonance frequency. The response is very small otherwise. You should know what a **resonance curve** looks like, and what it means.

The easy mechanical analogy is a kid on a swing. Pump legs (or get a little push) at just right timing intervals (frequency), and the swing really gets moving back and forth in a big arc. Pump the legs too fast or too slow, and go nowhere. The amplitude of the mechanical oscillation is akin to the voltage oscillation measured in the LC resonator.
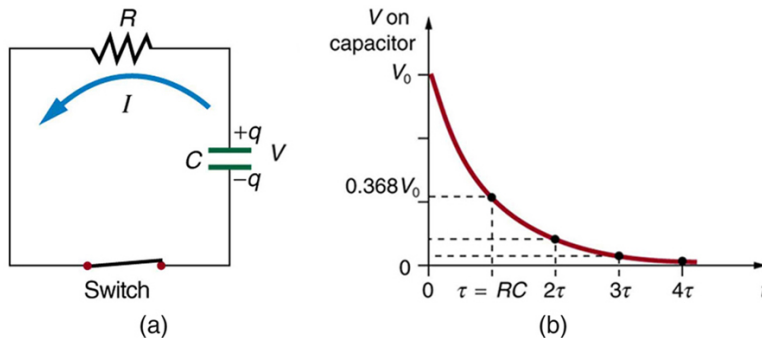
RC charging and discharging: Exponential in time

Figure 10: Parallel RC circuit with initial charge q_o on the capacitor with corresponding initial voltage v_o measured across it. Image credit: `https://openstax.org/books/college-physics/pages/21-6-dc-circuits-containing-resistors-and-capacitors`

A parallel RC circuit has a capacitor charged to some initial voltage v_o. The charge will run off the capacitor, through the resistor, where the energy is dissipated. The voltage measured across the capacitor and resistor parallel combo will exponentially decay as a function of time:

$$v(t) = v_o e^{-t/RC} = v_o e^{-t/\tau}.$$

Note that the **time constant** defined as $\tau = RC$ sets the decay time. If we plug in time $t = \tau$, then we get:

$$v(t = \tau) = v_o e^{-\tau/\tau} = v_o e^{-1} \approx 0.37 v_o.$$

So τ actually tell us the amount of time it takes for the voltage to fall to about 37% of the initial value. If τ is small, the capacitor quickly discharges; the larger τ the longer it takes for the capacitor to discharge. For example, if we start with $v_o = 5$V. and choose $C = 10$ nF and $R = 1$ kΩ, then in $\tau = 10$ μs, the voltage will be about 0.37×5V ≈ 1.8 V. If we chose a much larger resistor of $R = 100$ kΩ, then it would take $\tau = 1000\mu$s $= 1$ ms for the voltage to fall from 5V to 1.8 V.

Fluid analogy wise, think of a bath tub draining. The water level in the bathtub sets the extra pressure forcing the water to drain. This is akin to the charges on the capacitor creating extra "pressure" voltage causing charges to move. The size of the drain pipe is akin to the resistance. The smaller the drain pipe, the larger the resistance, therefore the longer it takes to drain the bath tub or capacitor.

In practical terms, a parallel combo of RC can be used to smooth out a sinusoidal varying signal. This "smoother" is commonly found in AC-DC converters and in AM radio receivers to recover the audio envelope from the AM Carrier wave.

Op-amps: Basic configurations

1. **Golden Rules:**

   Op-amps are complex beasts on the inside (40+ years of development), but we can analyze them in easy terms using two golden rules:

   1) $i_+ = 0; i_- = 0$. This just says there can be zero input current at the input of the op-amps. This rule applies no matter what (due to having gigaohms worth of resistance waiting just inside)

   2) $v_+ = v_-$. This rules applies only when the op-amp is in **negative feedback**. If so, the op-amp will work its little heart out to make sure the voltage at both input terminals remains equal.
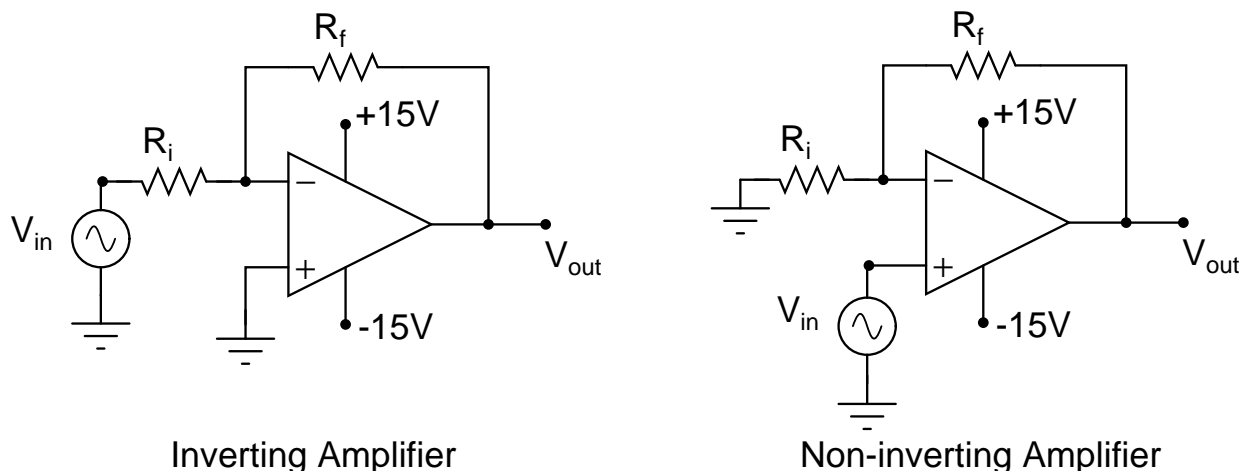

Inverting Amplifier   Non-inverting Amplifier

Figure 11: Basic op-amp configurations. In this example, the amplifier is powered by the dual supply ±15 V

2. **Inverting amplifier:** This is a basic config shown in Figure 11, left. The output signal is inverted (π out of phase) relative to the input and scaled by a factor that depends on the resistor values:

$$\frac{V_{out}}{V_{in}} = -\frac{R_f}{Ri} \tag{6}$$

   The factor of $\frac{R_f}{Ri}$ is typically referred to as the **voltage gain**, or more simply **gain** (not to be confused with decibel gain). Note the this amplifier can actually "deamplify": the output amplitude can be smaller than the input if $R_f < R_i$. This can be useful to reduce the amplitude of a very large signal.

3. **Non-inverting amplifier:** This is a basic config shown in Figure 11, right. The output signal is in phase with the input. The input-output relation is given by:

$$\frac{V_{out}}{V_{in}} = 1 + \frac{R_f}{Ri} \tag{7}$$

   Note the voltage gain is always at ≥ 1; no de-amplification is possible .
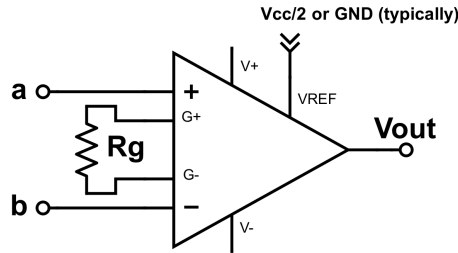
4. **Instrumentation amp:**

Figure 12: Basic instrumentation amp. Power supply connections not shown. V_{ref} is typically set to ground when using a dual power supply (e.g. $\pm$ 12 V), but is typically set to $V_{cc}/2$ for a single sided power supply (e.g. power connections are to $V_{cc} = 5$V and ground

Instrumentation amps (Figure 12) are widely used in many a project, anywhere you need to measure a difference between two points in a circuit. Wheatstone bridge for thermal sensing? Yep. EMG sensor to measure differences of voltage across two points on the skin. Instrumentation amp is your friend there too!

The instrumentation amp is a fancy difference amplifier:

$$V_{out} = G(V_{in}^+ - V_{in}^-) + V_{ref} \tag{8}$$

Equation 8 says that the instrumentation amp takes the difference between two voltages and amplifies them by a voltage gain G. Also, an offset "reference" voltage can also be added. This is very handy for single-supply operation.

The voltage gain can be set with a single resistor R_G (see Figure 12). Be aware the different models of instrumentation amps have different formulas for the voltage gain G. Some specific examples for G we use down in the lab include: the INA 126 which has a voltage gain $G = 5 + 80\text{k}/R_G$. Another common instrumentation amp in the lab is the AD623 (specifically designed for single supply, rail-to-rail operation), for which $G = 1 + 100\text{k}/R_G$. (Note the G used here is just voltage gain, not decibel gain...there are only so many letters in the alphabet, sigh).

5. **Buffer**

   The buffer (aka voltage follower) is used to isolate two functional blocks of circuitry. Essentially it reports a voltage without leaking current (thanks to op-amp input current = 0). Use it anytime you want to connect one functional block to another, without the risk of leaking current. For example, in Figure 13, the full voltage of the input source is present across the 100 Ω load (resistor). Without the buffer, the voltage across the 100 ohm would be only a fraction of the (1/11 in this case) of the input voltage; 10/11 is dropped across 1k resistor. We used a buffer in our EMG circuitry and vibration sensor to connect a voltage dividers to bias voltages of op-amps.
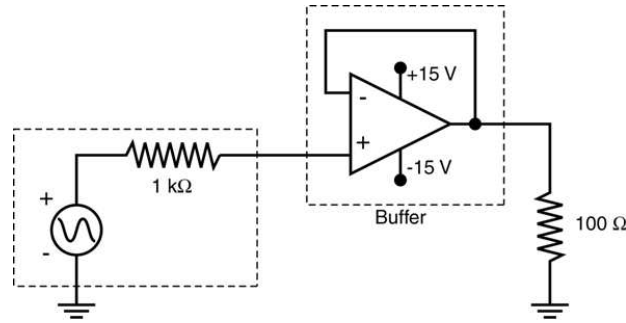

Figure 13: Op-amp buffer isolates source from a load. Power supply connections of $\pm$ 15V are suggestive only.

6. **Trans-impedance amplifier (TIA)** The job of the the TIA is to convert an current input into a voltage output. In the case of Figure 14, light incident on the photodiode generates a time-varying input current. The TIA correspondingly produced a changing voltage at the output. This configuration is also a close relative of an active-filter with a reference offset voltage (see Figure 15 [E]).
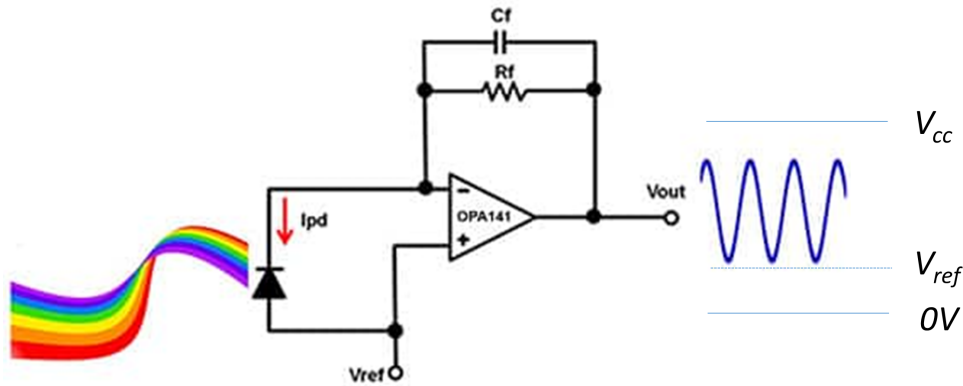

Figure 14: Trans-impedance amplifier with photodiode input. Image adapted from: Digikey Tutorial "How to Design Stable Transimpedance Amplifiers"

Active Filters

Active filters built around op-amps in an ***inverting amplifier configuration*** are illustrated in Figure 15. Active filters do 2 jobs at once: they filter and they amplify.
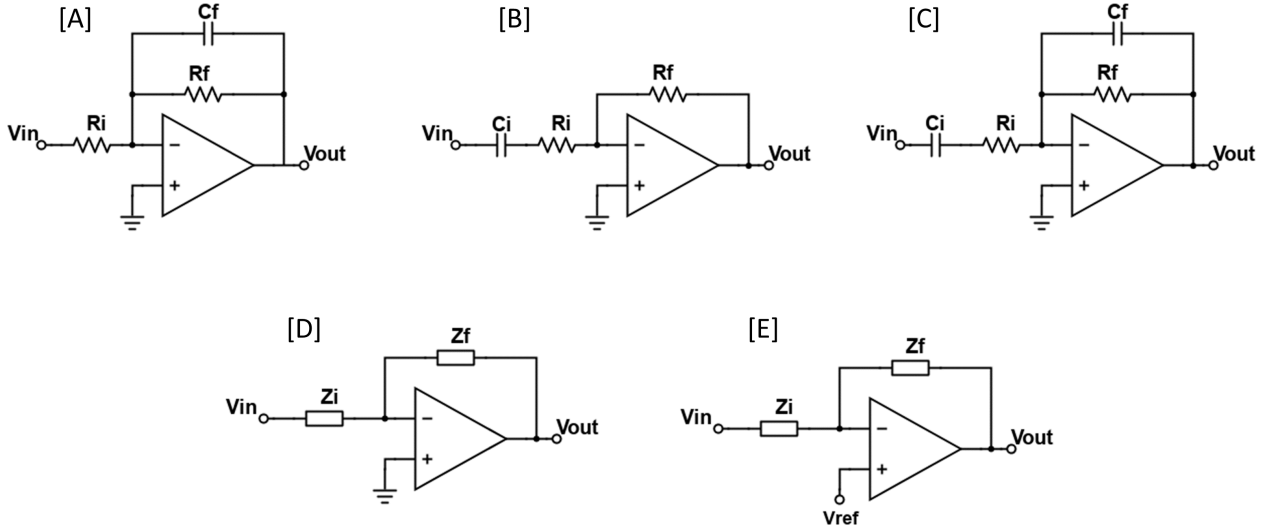

Figure 15: Active Filters (inverting op-amp config). [A]: LPF; [B]: HPF; [C]: BPF; [D]: Generic inverting amp configuration with non-inverting input connected to ground; [E] Typical single-supply design: inverting amp configuration with reference offset voltage.

1. **Generic inverting configuration [D]** Using golden rules it is easy to show the transfer function (input-output relation) is given by:

$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \tilde{H}(\omega) = -\frac{\tilde{Z}_f}{\tilde{Z}_i} \tag{9}$$

The key thing is to recognize what elements in an actual circuit constitute the feedback components modeled as an equivalent impedance $\tilde{Z}_f$, and which components constitute the input components modeled as $\tilde{Z}_i$.

2. **Active low pass filter [A]**

$$\tilde{H}_{LPF}(\omega) = \underbrace{-\frac{R_f}{R_i}}_{amplify} \underbrace{\left(\frac{1}{1 + j\omega R_f C_f}\right)}_{filter} \tag{10}$$

The transfer function can be easily decomposed into the amplification factor $-\frac{R_f}{R_i}$, and filtering section the passive low-pass filter transfer function. A similar statement can be made for all such active filters shown in this section.

3. **Active high pass filter [B]**

$$\tilde{H}_{HPF}(\omega) = -\frac{R_f}{R_i}\left(\frac{j\omega R_i C_i}{1 + j\omega R_i C_i}\right) \tag{11}$$

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4. **Active band Pass filter [C]**

$$\tilde{H}_{BPF}(\omega) = -\frac{R_f}{R_i}\left(\frac{1}{1 + j\omega R_f C_f}\right)\left(\frac{j\omega R_i C_i}{1 + j\omega R_i C_i}\right) \tag{12}$$

5. **Active filter without [D] and with reference offset [E]**

   In [D] the non-inverting terminal is grounded (reference voltage $= 0$). In [E], note the reference potential applied to the non-inverting terminal in circuit [E]. The reference potential is often used with ***single supply design*** to center the dynamic range of the output. Translation: Say we have a single supply system, an op-amp powered by $V_{cc} = 3.3$ V and ground (in contrast with a dual supply system powered with $\pm$ 3.3 V). We can center the baseline output to $V_{cc}/2 = 1.65$ V by connect V_+ to a constant $V_{ref} = V_{cc}/2$. Consider 3 cases:

   (a) Input signal is zero: $v_{in}(t) = 0$. Then the output will hold steady at $v_{out}(t) = V_{cc}/2 = 1.65$ volts.

   (b) Input signal is positive: $v_{in}(t) > 0$ (e.g. top half of sine wave) The inverting amp will corresponding decrease below 1.65 V, reaching as low as ≈ 0 V (with a ***rail-to-rail*** op-amp)

   (c) Input signal is positive: $v_{in}(t) < 0$ (e.g. bottom half of sine wave). The inverting amp will corresponding increase above 1.65 V, reaching as high as $\approx V_{cc}$, 3.3 V in this case.

   Note by centering the op-amp that both positive and negative wiggles in the input can be properly represented at the output with corresponding wiggles up and down. By contrast, if set the reference voltage to be zero (e.g. grounded the non-inverting input), the baseline output would sit at 0 V. The output voltage can't go any lower than this, so the positive wiggles at the input wouldn't properly appear amplified and filtered at the output. Instead, they would be ***clipped***.

   In fancy math terms, we can write this as:

$$\tilde{V}_{out} = -\frac{\tilde{Z}_f}{\tilde{Z}_i}\tilde{V}_{in} + \left(1 + \frac{\tilde{Z}_f}{\tilde{Z}_i}\right)V_{ref} \tag{13}$$

   Note the right hand term can be expanded to: $V_{ref} + \frac{\tilde{Z}_f}{\tilde{Z}_i}V_{ref}$. Thus, to achieve a baseline output of V_{ref}, we must rely on the filter action (encoded in $\frac{\tilde{Z}_f}{\tilde{Z}_i}$) to reject a constant voltage ($\omega = 0$). Only the HPF and BPF do so.

   Note if the reference is set to 0 (connected to ground), we recover the familiar transfer function $\tilde{H}(\omega) = -\frac{\tilde{Z}_f}{\tilde{Z}_i}$.

6. **Cascading Active Filters**

   Think of these active filters as functional blocks. We can freely cascade them together *without* needing to obey the $10\times$ rule that we must follow for passive RC filters. Op-amps are active elements, and each one in the cascade will work their little hearts out to make sure the input-output relationship is obeyed for each functional block. Hence, no worries (to first approximation) about current leakage in successive cascaded stages that was our concern with passive RC filters.

14

Diodes/Semiconductors

Semiconductor physics:

1. n-type and p-type doped materials

2. PN junction

3. diode curves for different materials (silicon vs. red LED vs. blue LED, etc.)

4. forward/reverse bias in diodes

5. photodiode operation

6. MOSFET digital switches

 Applications:

1. LED practical: current limiting resistors

2. Diode drops is material property: ≈ 0.7 V for standard Si diode, ≈ 1.6 V for red (GaAs) etc)

3. Rectifiers

4. Peak detector

Arduino Programming

Practical stuff you should feel comfortable doing includes:

1. Reading analog voltage with the analog to digital converter (ADC)

2. 10-bit digital conversion voltage

3. Turning on or off digital pins to illuminate LEDs, for instance

4. Basic math computations (addition, multiplication, etc.)

5. Digital pulses (PWM) to control servo motors and sound buzzers

6. Text display in the serial monitor

7. Graphical display in the serial plotter