Concept Inventory
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Foreword

We've seen that the wave equation is used to describe a vast array of physical phenomenon, from traveling waves on guitar and piano strings, to optical waves, electromagnetic waves, sound waves, and even the strange quantum world. Our goal here is to solve the 1-D wave equation using ***Separation of Variables*** (SoV). In essence, SoV breaks down a Partial Differential Equation (PDE) into familiar Ordinary Differential Equations—which we already know how to solve! We'll start with just 1-D, but note this technique generalizes to multiple dimensions as well! Lastly, note that the SoV technique is entirely general and can also be employed to solve the Diffusion Equation too! The SoV: it slices, it dices...orders yours now within 30 min and get 2 free to share with friends :)

Separation of Variables: Conceptual roadmap

1. Start with the 1-D wave equation:
$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$
Note our function $u(x,t)$ is a function of space *and* time. Eqn 1 is called a ***partial differential equation (PDE)*** simply because partial derivatives are involved.

   We are going to *assume* the following boundary conditions, namely that our physical quantity of interest (displacement, pressure, whatever) is fixed at both ends of our domain running from $0 \le x \le L$:
$$u(0,t) = 0;\ u(L,t) = 0 \tag{2}$$
Note that the actual boundary conditions will depend on the problem context. We might have fixed ends, or we might have free ends or a mix (e.g. a cable that is fixed at one end, free at the other).

2. Take a leap of mathematical faith: make a (very good) guess that we we can write our solution $u(x,t)$ as follows:
$$u(x,t) = X(x)\,T(t) \tag{3}$$
Here, we have *separated* our variables for space x and time t. The function $X(x)$ carries all the information about how our physical variable of interest (displacement, pressure, whatever) varies in space; whereas the function $T(t)$ carries all the information about our our physical variable varies with time.

3. Plugging our guess for $u(x,t)$ back into Eqn 1, show that we can write:
$$\frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \tag{4}$$
Note that in Eqn 4 we have ***fully separated our variables*** for time and space. The LHS has only to do with time; the RHS only with space.

4. Now for something a bit subtle, but very important. Justify why can massage Eqn 4 into the following form, ***replacing partials for full derivatives***:

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} \tag{5}$$

5. Now for the most difficult conceptual piece of the SoV puzzle. We are going to add something very subtle to Eqn 6 as follows:

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \tag{6}$$

   where k^2 is assumed to be a positive real-valued constant. Thus $-k^2$ is negative valued and real. Two magic math moments to contemplate and justify here:

   (a) Why can we set both LHS and RHS equal to a constant???

   (b) Why did we (rightly/safely) assume *for this particular problem* that $-k^2 < 0$ and real-valued? (This may become more obvious in the next step)

6. Let's now solve for $X(x)$. Isolate just the RHS equality of Eqn 6 and show we can write the ODE for our spatial eigenfunctions $X(x)$

   Remember, X is only a function of our spatial coordinate x.

$$\frac{d^2X}{dx^2} = -k^2X \quad \text{or} \quad \frac{d^2X}{dx^2} + k^2X = 0 \tag{7}$$

   This is hopefully starting to look mighty familiar!

7. Next up: solve the ODE in 7. You can leave the parameter k in your solution for now. And remember: a 2nd order ODE has *two* functions that solve it. Also, experiential wisdom suggests it is nicer here to start writing sin and cos instead of complex exponentials (in case you went that route).

8. Are both of the functions from the previous step permissible? And what values of k are permissible? As of now, k is just some magical math constant (often termed the ***separation constant***). To solve these mysteries of the math universe, namely to determine permissible values for k, we need to apply ***boundary conditions***.

   As stated before, we are going to *assume* the following boundary conditions are in play here, namely that our physical quantity of interest (displacement, pressure, whatever) is fixed at both ends (think: guitar string) of our domain running from $0 \leq x \leq L$:

$$u(0,t) = 0; \; u(L,t) = 0 \tag{8}$$

   Refine the solution for $X(x)$ and clearly state permissible values of k_n which meet this boundary conditions. Hint: They are now cleverly indexed with n; you should get infinitely many of them in integer multiples. And only one of the 2 functions should survive the day.

9. We now have solutions for our spatial functions $X_n(x)$ in hand, hooray! Again, note the subindex n keeps track of which spatial ***mode*** we are talking about. Sketch the first 3 of them ($n = 1, 2, 3$).

10. We're more than half-way there. We're more than livin' on a math prayer (you had that right, Bon Jovi). We just have to circle back to solve for our function of time $T(t)$.

    Recall back at the ranch (Eqn 6) that we had the following:

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2 \tag{9}$$

    Rearrange this into another very familiar form:

$$\frac{d^2T}{dt^2} = -c^2k_n^2T \quad \text{or} \quad \frac{d^2T}{dt^2} + \omega_n^2T = 0 \tag{10}$$

    Take careful note of how we defined the temporal angular frequency ω in terms of the spatial frequency parameters k and the propagation velocity c. Note the subindex n now in play, which carries through from our previous work solving $X_n(x)$ in the steps above.

    How does ω_1 compare to say ω_3? Which mode $n = 1$ or $n = 3$ will oscillate up and down faster? How can you be sure/justify? (Recall, c is assumed to be a constant in this problem.)

11. Now solve the ODE in Eqn 10, which hopefully looks like mighty familiar to you (from our previous work in class and prob set 3!) Again, recall that a 2nd order ODE has two functions that solve it. Again, probably better/more intuitive here to write trig than complex exponentials, but either will do. You should now have the time (only) dependent functions in hand $T(t)$. Hooray!

12. Sketch the first 3 of these $T_n(t)$ vs. time ($n = 1, 2, 3$). Again, n indexes which ***mode*** we are talking about.

13. The last big conceptual step we'll take here is this: You should have found a whole family of functions $u_n(x,t) = X_n(x)\,T_n(t)$ that solve Eqn 1. One final reminder: they carry subscripts n denoted the nth ***mode***.

    Our last conceptual leap of faith is this: any and all of these modes solve the 1-D wave equation. That implies a ***superposition*** of them also solves the PDE (1).

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) \tag{11}$$

    Thus, justify that we can write the total solution for this problem as:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(k_n x)\left[a_n \cos\omega_n t + b_n \sin\omega_n t\right] \tag{12}$$

14. At this point, your keen eye is probably scrutinizing Eqn 12 and thinking "Hmmm, looks mostly ok, but we don't know what a_n and b_n are yet. Bingo! It turns out these are ***Fourier coefficients*** in disguise. Remember Fourier? Of course, you do! :)

    These coefficients must be solved using ***initial conditions***. In a typical problem, we are given the starting parameters, namely the shape of the string and velocity of the string along its entire length:

$$u(x,0) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0)$$

For example, we might have initial velocity along the entire length is 0 and the initial shape is given a triangular shape as shown here (link).

Specific to these initial conditions:

(a) Make a quick and convincing argument why all the $b_n = 0$

(b) Plugging in $t = 0$ show that we just have a Fourier series:

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(k_n x) \tag{13}$$

This just says we can synthesize any arbitrary initial shape by summing up a bunch of sin functions each of which mixes in amplitude a_n.

(c) Detail how you would solve the a_n just like we did with Fourier series.

15. For now, we'll let the math pen rest, as the main thrust of this workshop was really just to get you used to the *process* of *separation of variables*. The curious math cat may continue on to solve out for a_n (hello, final problem!). For now, gain some intuition by reviewing solutions or plucked stringed instruments (guitars) and struck stringed instruments (pianos). Relate these in your minds eye back to all the math we just did! Being able to see and understand the individual modes in action in super powerful. Once you see what's going, writing the solution is "just minor math details" (famous last words).

Recap

Let's recap what we just did:

1. We just made a good guess that our total solution could be written as a ***product*** of function in space and time: $u(x, t) = X(x)T(t)$. For example, $\sin k_n x \cos \omega_n t$ qualifies, whereas $\sin xt$ does not.

2. We discovered the PDE boils down to two ODEs. Woohoo, we know how to solve these. It should come as no surprise that the wave equation kicks out a solution of sines and cosines.

3. For the spatial functions $X(x)$ we carefully applied boundary conditions to find just what modes were allowed, setting the exact shape of the sines and cosines that solve the problem at hand.

4. At last we returned to get our temporal functions $T(t)$, realizing that the frequency of oscillation is inextricably linked to the mode shape. Generally, the higher order the mode, the faster the oscillations.

5. We summed infinitely many sines and cosines; all of the possible modes can contribute to the total solution!

6. Penultimately, we applied initial conditions and saw how Fourier series comes into play to fully solve out the Fourier coefficients!

7. Finally, and perhaps most importantly, we linked the math back to visualizations. The latter are so powerful for training your math brain!