ODEs: A quick intro to unforced solutions
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Foreword

Many times in many branches of STEM, you will find yourself needing to solve a 1st or 2nd order. There are 2 main solution strategies that we will employ:

1. Separation of variables
2. Guess and check! This works because so many physical systems either oscillate in time, show real exponential behavior in time (e.g. exponential decay); or both. Hello again, complex exponentials

ODEs entree

1. Here's a classic example of a ***1st order ordinary differential equation*** (ODE). 1st order refers to the fact that the highest order derivative below is the first derivative dn/dt. "Ordinary" simply means there are no partial derivates involved.

   Let's say we are interested in number of volatile organic air molecules as a function of time $n(t)$. These molecules undergo a chemical reaction (e.g. when exposed to sunlight) where they break down, leaving fewer molecules than before. We can fairly claim that the change in number of molecules is proportional to the number of current molecules. The rate of this decay is given by positive,real constant $\alpha > 0$. Thus we can write:

$$\frac{dn}{dt} = -\alpha n \tag{1}$$

   Let's also stipulate an ***initial condition*** of how many molecules exist at time $t = 0$, $n(0) = n_o = 10^9$.

   (a) Write an ***analytical solution*** for $n(t)$ that solves Eqn 1 AND adheres to the initial condition.
   (b) Sketch your solution for $\alpha = 1/10$, 1, and 10. (Note: units of α must be number of molecules per unit time.Assume here that the units are number per ms.)
   (c) In each case, how much time must elapse until half the original population of molecules remain. (This is typically called the ***half-life***.)
   (d) How would the nature of the solution change if our ODE was instead $\frac{dn}{dt} = +\alpha n$?

2. Now for a 2nd order ODE. This is called 2nd order because the second derivative is the highest involved derivative involved. A simple mechanical system modeled as a single mass on a spring is described by the 2nd order ODE:

$$m\ddot{x} + kx = 0. \tag{2}$$

   We'll see in a second that the solutions for $x(t)$ are just oscillations in time, hence this model is termed the simple harmonic oscillator (SHO). The SHO is elegant in its simplicity, yet is a good first order approximation of many systems throughout science and engineering: chemical bonds oscillate; civil structures such as building and bridges oscillate; electrical charge oscillates in a audio devices and mobile phone antennas; air molecules oscillate inside a wind instrument; and so on. For now, let's keep the SHO simple and intuitive. Let's just let it be a mass on a spring.

Given mass $m = 6$ kg and spring constant $k = 24$ N/m, and the initial condition $\dot{x}(0) = 5$ cm/s. and $x(0) = 10$ cm, write an expression for $x(t)$ and sketch your solution. Try two solution methods. They should work equally well—two sides of the same math coin, so to speak.

(a) Solution method 1: You know what the answer has to be physically. The system oscillates. So the solution MUST involve sines and cosines. Note that we actually need both here to account for TWO initial conditions.

(b) Solution method 2: Let's assume we are naive and don't really know the nature of the system. One guess that almost always works is: $x(t) = de^{rt}$. The trick here is to find the characteristic roots r that actually solve Eqn 2.

3. Now modify the mechanical system to include some damping. Damping dissipates energy from a system. Sometimes this is intentionally part of the design: Cars have shocks to dissipate energy so you don't oscillate forever going over a speed bump. Damping is often modeled using first order derivative term and modifying the SHO equation to create a ***damped oscillator*** as follows:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3}$$

(a) Find an expression for $x(t)$ that solves Eqn 3. Hint: Again, just guess γe^{rt} and solve for the characteristic roots.

(b) Let the damping constant $c = 10$ Ns/m. Given the same mass, spring constant, and initial conditions as above, write an expression for $x(t)$ and carefully sketch your solution.

(c) Let's take this damped oscillator as a model for a mountain bike suspension system going over a bump or landing a big jump (or it could be an auto going over a speed bump, dune buggy dune hopping, you name it). Based on your answer in part b, estimate the how amount of time for which the rider feels "significant" up and down motion after having gone over bump or landing a jump. How many oscillations will have occurred during until that "settling time"?