SOLUTIONS to ODEs: A quick intro to unforced solutions
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ODEs entree

1. Here's a classic example of a ***1st order ordinary differential equation*** (ODE). Let's say we are interested in number of volatile organic air molecules as a function of time $n(t)$. These molecules undergo a chemical reaction (e.g. when exposed to sunlight) where they break down, leaving fewer molecules than before. We can fairly claim that the change in number of molecules is proportional to the number of current molecules. The rate of this decay is given by positive, real constant $\alpha > 0$. Thus we can write:

$$\frac{dn}{dt} = -\alpha n \tag{1}$$

Let's also stipulate an ***initial condition*** of how many molecules exist at time $t = 0$, $n(0) = n_o = 10^9$.

(a) Write an ***analytical solution*** for $n(t)$ that solves Eqn 1 AND adheres to the initial condition.

We'll formally solve Eqn 1 below. Before we do, stop and think for a moment what this equation actually says: Find a function $n(t)$ where we take a derivative with respect to time (LHS) which returns the function itself times a constant (RHS). How many functions can you name that meet this criterion? Hopefully, the exponential comes immediately to mind. Indeed, this is the solution.

Now let's use separation of variables to show this formally—or assuming we didn't have the insight above. Starting with:

$$\frac{dn}{dt} = -\alpha n$$

we need to get all of the n's on one side and all of the t's on the other:

$$\frac{dn}{n} = -\alpha dt$$

Then we can integrate both sides:

$$\int \frac{dn}{n} = \int -\alpha dt$$

These are indefinite integrals, so we include the integration constants A and B here to get:

$$\ln n + A = -\alpha t + B$$

Combine the integration constants $C = B - A =$ to arrive at:

$$\ln n = -\alpha t + C$$

Then we need to solve for $n(t)$. To unlock this, take the exponential of both sides:

$$e^{\ln n} = e^{-\alpha t + C}$$

Using the property $e^{a+b} = e^a e^b$ we can simplify to:

$$n = e^{-\alpha t} e^C$$

Note e^C is just some constant (since C is constant), which we will call D. Thus:

$$n(t) = De^{-\alpha t}$$

Here we have written $n(t)$ to explicitly denote it is a function of time. What is D? This is where we need to use the initial condition. We know that $n(t = 0) = n_o$.

Plug $t = 0$ into our equation for $n(t)$ above to get that $D = n_o$ (because $e^0 = 1$).

Thus, our solution we've been searching for is:

$$n(t) = n_o\, e^{-\alpha t}$$

It's a decaying exponential! Remember way back at the ranch when we our insight led us to think that $n(t)$ must be an exponential because the ODE said "take a derivative and you'll get back the function times a constant". This is the only function the meets this criterion, so no surprise. Moral of the story: insight is more incisive and efficient than brute force.

Lastly, one important note. There is one and only one function that solves this first order ODE in Eqn 1. And we required only a single initial condition to solve it completely. We will not prove it here, but first order ODEs only have a single solution. In the next section, we will deal with a second order ODE and find there are not one but two functions that solve the equation. We will also require two initial conditions to fully solve it In general, an mth order ODE has m linearly independent solutions and requires m initial conditions to completely solve. We'll save the proof for another day (aka math 332).

(b) Sketch your solution for $\alpha = 1/10$, 1, and 10. (Note: units of α must be number of molecules per unit time.Assume here that the units are number per ms.)

Note that the value of α sets the time scale. Lower values of α mean it takes longer for the function to decay. That makes intuitive sense because the rate of decay dn/dt is proportional to alpha per Eqn 1.

Another way to see it: let's ask how much time must elapse until we reach $e^{-\alpha t} = e^{-1} \approx 0.37$. When $\alpha = 1/10$ it takes 10 units of time for this to happen (10 ms here, per problem statement). By contrast, when $\alpha = 10$, only 1/10 ms must elapse. This is much faster decay. It's left as an exercise to the reader to sketch $n(t)$ for the various values of alpha.

(c) In each case, how much time must elapse until half the original population of molecules remain. (This is typically called the ***half-life***.)

Here we just set $n(t = t_{1/2}) = n_o/2 = n_o e^{-\alpha t}$. Then solve for the half life:

$$\frac{1}{2} = e^{-\alpha t_{1/2}}$$

Thus, $t_{1/2} = -\frac{\ln(1/2)}{\alpha} = \alpha \ln 2$.

In the last step we have used the fact that $-\ln\frac{1}{2} = \ln(\frac{1}{2})^{-1} = \ln 2$.

(d) How would the nature of the solution change if our ODE was instead $\frac{dn}{dt} = +\alpha n$?
Just a minus sign, right? Well, this changes everything! Instead of a decaying exponential we will now have a positive exponent, thus the solution for $n(t)$ blows up exponentially in time according to:

$$n(t) = n_o \, e^{\alpha t}$$

This is what happens when bunnies get together. Repeatedly.

2. Now for a 2nd order ODE. This is called 2nd order because the second derivative is the highest involved derivative involved. A mechanical system is described by the 2nd order ODE:

$$m\ddot{x} + kx = 0. \tag{2}$$

Given $m = 6$ kg and $k = 24$ N/m, and the initial condition $\dot{x}(0) = 5$ cm/s. and $x(0) = 10$ cm, write an expression for $x(t)$ and sketch your solution. Try two solution methods. They should work equally well—two sides of the same math coin, so to speak.

   (a) Solution method 1: You know what the answer has to be physically. The system oscillates. So the solution MUST involve sines and cosines. Note that we actually need both here to account for TWO initial conditions.

   Before blazing through a solution, let's stop and think for a second what Eqn 2 actually says. Let's first rearrange slightly:

$$\ddot{x} = -\frac{k}{m}x \tag{3}$$

   This says we must find a function $x(t)$ where we take a two derivatives with respect to time (LHS) which returns the function itself times some constants out front (RHS). How many functions have this special property? Hopefully, you'll think of three: sine, cosine, and an exponential. These are all valid solutions as we will see.

   Let's start with a (very good) guess of sines and cosines for our solution as follows:

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t \tag{4}$$

   Note that we need to solve for the angular frequency ω as well as the constants C_1 and C_2. The former will be solved using the ODE Eqn 3. The constants out in front will be solved using our two initial conditions.

   First, let's check whether our guess for $x(t)$ satisfies the ODE in Eqn 3. Just take two derivaties with respect to time. This kicks out a factor of ω^2 out front of both the cosine and sine term. Also, we get a negative sign out front taking two derivatives of the cosine and sine function. All together this yields:

$$\begin{aligned} \ddot{x}(t) &= -\omega^2 C_1 \cos \omega t + -\omega^2 C_2 \sin \omega t & (5) \\ &= -\omega^2 \underbrace{(C_1 \cos \omega t + C_2 \sin \omega t)}_{x(t)} & (6) \\ &= -\omega^2 x(t) & (7) \end{aligned}$$

   Wow, mathemagic just happened! We took two time derivatives of our guess for $x(t)$ given by Eqn 4 and got back the function itself times some (real) constant ω^2 with a minus sign out front. Just as the math oracle foretold—or hoped for—in view of the ODE Eqn 3.

   We can peer back at Eqn 3 and instantly realize that can relate the the angular velocity (how fast something oscillates back and forth) to the mass and spring constant:

$$\omega^2 = \frac{k}{m}$$

Hooray—one down, two to go. We still need to solve for constants C_1 and C_2. We'll use initial conditions to solve those.

Let's pause to realize that we have two functions that solved the 2nd order ode—both the sine and the cosine. We need both because one allows for non-zero initial position and the other allows for non-zero initial velocity.

To solve for C_1, we can use the fact that $x(0) = x_o$. Plug in $t = 0$ to get:

$$x(0) = C_1 \underbrace{\cos 0}_{1} + C_2 \underbrace{\sin 0}_{0}$$

Thus, we can easily identify that

$$x(0) = C_1 = x_o$$

Now for C_2 we need to take a derivative of $x(t)$ and plug in $t = 0$:

$$\dot{x}(0) = \dot{x}_o = v_o = -\omega C_1 \underbrace{\sin 0}_{0} + \omega C_2 \underbrace{\cos 0}_{1}$$

From here it is easily to identify that:

$$C_2 = \frac{v_o}{\omega}$$

Ta-dah! We've got everything we need to write the full solution. We now have expressions for ω, C_1 and C_2 in our solution for $x(t)$ given by Eqn 4.

It is left as an exercise to the reader to plug in numbers and find that $\omega = 2$ rad/s; $C_1 = 10$ cm; and $C_2 = 5/2$ cm.

Lastly, note that the ODE we started with is often called the ***simple harmonic oscillator***, commonly written as:

$$\ddot{x} + \omega^2 x = 0 \tag{8}$$

It appears in pretty much every branch of engineering and physics, because everything can be modeled, to first approximation, as a mass on a spring.

(b) Solution method 2: Let's assume we are naive and don't really know the nature of the system. One guess that almost always works is: $x(t) = ce^{rt}$. The trick here is to find the characteristic roots r that actually solve Eqn 3.

This is left as an exercise to the reader. Just plug in $x(t)$ to Eqn 3 then find out what the roots r must be in order to satisfy the original ODE.

$$\ddot{x}(t) = r^2 \underbrace{ce^{rt}}_{x(t)}$$

This looks promising so far because taking two derivatives wrt time kicked back the function $x(t)$. Let's see what we have now, plugging in our expression above into the simple harmonic oscillator ODE Eqn 8:

$$r^2 ce^{rt} + \omega^2 ce^{rt} = 0$$

Rearrange and find that:

$$(r^2 + \omega^2)ce^{rt} = 0$$

There are 3 options to make this equation true:

(a) $c = 0$. This is boring. This makes $x(t)$ zero for all time. Nothing actually oscillates! This is called a trivial solution and we discard it here. We want to bounce around the room.

(b) $e^{rt} = 0$ for all time. Impossible, can't happen.

(c) $r^2 + \omega^2 = 0$. This could actually be the case! Let's follow this path. It implies that $r = \sqrt{-\omega^2} = \sqrt{-1}\sqrt{\omega^2} = \pm i\omega$. We call r the ***characteristic roots***. Note we accept both of these roots r for our solution. Let's continue this path....

So now we know our two functions that solve the ODE. Which is to say:

$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \tag{9}$$

The complex exponentials are really just hiding sines and cosines. This becomes apparent if we use Euler's ID:

$$x(t) = c_1(\cos\omega t + i\sin\omega t) + c_2(\cos(-\omega t) + i\sin(-\omega t))$$

We can recognize cosine is even ($\cos(-x) = \cos x$) and sine is odd ($\sin(-x) = -\sin x$) such that we can write:

$$x(t) = c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t)$$

Careful here with c_1 and c_2. They can be–and in fact are—complex numbers. E.g: $c_1 = a_1 + i\,b_1$ and $c_2 = a_2 + i\,b_2$. Physically this MUST be the case because oscillations are a real physical quantity, not just some crazy abstract math. In fact, we'll see shortly that c_1 and c_2 are complex conjugates. How to solve for them? We need to apply our initial conditions, just as before!

Let's plug $t = 0$ into our solution for $x(t)$ given in Eqn 9.

$$x(0) = c_1 e^0 + c_2 e^0$$

Therefore

$$c_1 + c_2 = x_o$$

Similarly, take a time derivative and plug in $t = 0$ for the second initial condition

$$\dot{x}(0) = c_1 i\omega e^0 + c_2(-i\omega)e^0 = \dot{x}_o$$

Thus, we get:

$$c_1 - c_2 = \frac{\dot{x}_o}{i\omega} = -i\frac{\dot{x}_o}{\omega}$$

where the last relation follows from multiplying top and bottom by i/i, a special form of 1.

We now have a system of 2 equations for 2 unknowns, namely c_1 and c_2. It is easy to solve this system to show that:

$$c_1 = \frac{1}{2}(x_o - i\frac{\dot{x}_o}{\omega}) \quad \text{and} \quad c_2 = \frac{1}{2}(x_o + i\frac{\dot{x}_o}{\omega})$$

Remember, we know the actual values for the initial position and velocity, so we're done! Note that these are complex conjugates, as previously claimed they would be. Mathemagic!

You might peer up at the expression for our solutions for $x(t)$ and c_1 and c_2 and ponder: *This is totally crazy—we have all these complex numbers running around with imaginary parts, but the solution is real valued to represent a real physical quantity?!*

The answer turns out to be yes, it is a real quantity! Again, the complex exponential are just sines and cosines hiding in disguise. The dear reader might also consider drawing a pair of complex conjugates, i.e. take $1 + 2i$ and $1 - 2i$, then add them up. What do you get? A real number! That seems promising. Which way do $e^{\pm i\omega t}$ rotate in the complex plane? What guarantees that they always sum up to a number on the real axis? Therein lies the key to fully understanding this math craziness.

3. Now modify the mechanical system to include a first order derivative term: $m\ddot{x}+c\dot{x}+kx=0$. Let $c = 10$ Ns/m. Given the same initial conditions as above, write an expression for $x(t)$ and carefully sketch your solution.

   All righty, let's make an informed guess that $x(t) = \gamma\, e^{rt}$. Note we are using γ to denote the constant out front, since we already have a c in the problem representing how much damping we have in the system. We are on the hunt for values of r and γ that (might) solve the ODE. We know to expect *two* characteristic roots, such that we get two functions that solve this 2nd order ODE, i.e., the full solution will look like: Remember, the order of ODE tells you how many functions solve it.

   The first time derivative is given by:

$$\dot{x} = \frac{d\left(\gamma e^{rt}\right)}{dt} = r\,\underbrace{\gamma e^{rt}}_{x(t)} = rx(t) \tag{10}$$

   Similarly, we can easily find the second time derivative:

$$\ddot{x} = r^2\,\underbrace{\gamma e^{rt}}_{x(t)} = r^2 x(t) \tag{11}$$

   Now we plug these back into the original ODE:

$$m\ddot{x} + c\dot{x} + kx = mr^2 x(t) + crx(t) + kx(t) = 0$$

   This can be rearranged pulling out the common factor of $x(t)$:

$$\left(mr^2 + cr + k\right) x(t) = 0$$

   Clearly, we don't want x(t) to be 0. That's the (boring) trivial solution where nothing actually oscillates. The other solution requires that the term in parenthesis equals 0. THis is a quadratic equation in r In other words, our guess for the solution to $x(t)$ which will perfectly well for special values of r called the ***characteristic roots*** (so called because they solve the quadratic equation:

$$mr^2 + cr + k = 0$$

   Writing out the quadratic formula we find that the characteristic roots are given by:

$$\begin{aligned} r &= \frac{-c \pm \sqrt{c^2 - 4km}}{2m} && (12)\\ &= -\frac{c}{2m} \pm \sqrt{(c/4m)^2 - k/m} && (13)\\ &= -\frac{c}{2m} \pm \sqrt{-1\left(\frac{k}{m} - \left(\frac{c}{4m}\right)^2\right)} && (14)\\ &= -\frac{c}{2m} \pm i\sqrt{\left(\frac{k}{m} - \left(\frac{c}{4m}\right)^2\right)} && (15)\\ &\approx -0.83 \pm i\,1.96 && (16) \end{aligned}$$

In this last step, we pulled out the factor of i with a bit of math prescience (experiential wisdom!), such that we have ***complex conjugates*** for our solutions. We can write these as

$$r_{1,2} = -\alpha \pm i\omega$$

with $\alpha = 0.83$ and $\omega = 1.96$ Intuitively, we guessed we'd have damped oscillations in our system, which requires a solution e-to-the-i-something (Old McDonald's song was originally about complex exponential, BTW), and now we see it coming to fruition.

Specifically, we can write our solution for the displacement vs. time:

$$\begin{aligned} x(t) &= \gamma_1 e^{r_1 t} + \gamma_2 e^{r_2 t} && (17)\\ &= \gamma_1 e^{(-\alpha+i\omega)t} + c_2 e^{(-\alpha-i\omega)t} && (18)\\ &= \underbrace{e^{-\alpha t}}_{damping} \underbrace{\left(\gamma_1 e^{i\omega t} + \gamma_2 e^{-i\omega t}\right)}_{oscillations} = \underbrace{e^{-\alpha t}}_{damping} \underbrace{\left(A\cos\omega t + B\sin\omega t\right)}_{oscillations} && (19)\\ & && (20) \end{aligned}$$

The last step is left to the reader as an exercise.

Now it is easy to see we oscillations which are multiplied by a decaying exponential, which decreases their amplitude over time. This is damped oscillations! The value of α controls how fast the system damps (as in problem 1!) and the value of ω controls how fast the system oscillates (1.96 rad/s, in this case). This is what we called ω in problem 2!

The remaining unknowns are the constant A and B.

The solution strategy is the same as was before; the procedure is straight-forward but requires careful attention to detail, especially applying the condition for initial velocity, wherein one uses the product and chain rules.

It is not too difficult, and left as an exericse to the reader to show that:

$$A = x_o$$

and

$$B = \frac{\dot{x}_o + \alpha x_o}{\omega}$$

.

Note that the amoun the sin term mixes in is dependent on initial velocity and position, while the cos term depends on initial position only. This isn't too starkly different from the simple harmonic oscillator (no damping).

Before we bid *adieux* to the study of this solution, we would should note that we were not a prior *guaranteed* to get damped oscillations. Our key insight was that term inside the square root $\left(\frac{k}{m} - \left(\frac{c}{4m}\right)^2\right) \approx 3.83$ is *positive* for the given values of m, c, and k. Most real-world systems fall into this case—whether mechanical, electrical, acoustic, etc—and are said to be ***underdamped***. Note this didn't necessarily have to be the case; we could have chosen mechanical parameters such that the value inside the square root would be negative which would make both characteristic roots real valued, thus there are two exponentially decaying solutions instead of damped oscillations. Lastly, note if we get rid of the damping ($c = 0$), we just recover our earlier solution for a simple harmonic oscillator with roots of $\pm i\sqrt{k/m}$.

There we have it, we've found a solution to our damped oscillation system! To recap, all we really did is:

(a) Make a good guess for of γe^{rt}, plug into the ODE and find out the special values of r that solve the ODE.

(b) Then we plugged in initial conditions.

That's it! Not too bad, dare we say "easy", right? ODEs are hopefully not so scary after all :)