Fourier Series workshop
ENGN/PHYS 225—Winter 2021 (March 26, 2021)

Foreword

This little workshop dives into the world of Fourier synthesis and analysis. Choose to tackle which ever problems interest you, possibly influenced by the real world context. Either way, the underlying math is the same!

Word Problems

1. **Symphony in C**

   In class, we mentioned how complex exponentials version of the Fourier Series can be used to represent a $2T$ periodic function $f(t)$ as follows: can be used to represent a Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{T}t} \tag{1}$$

   The crux of Fourier Series analysis is to find the c_n values specifying how much the nth harmonic contributes to the function $f(t)$. We threw up on the board the following formula for to find these complex Fourier coefficients:

$$c_n = \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\frac{n\pi}{T}t}dt \tag{2}$$

   There was a quick mention of "You can show this is actually the case using ***orthogonality*** of complex exponentials. Now it is time to dig into the details and show this is actually the case.

   Two functions $f(t)$ and $g(t)$ are said to be orthogonal if:

$$\frac{1}{2T}\int_{-T}^{T} f(t)g^*(t)dt = 0 \tag{3}$$

   where the $*$ denotes the complex conjugate. This integral is the continuous function equivalent of saying that two N-dimensional vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \ldots a_Nb_N = 0$.

   (a) Show that the functions $f(t) = e^{i\frac{n\pi}{T}t}$ $g(t) = e^{i\frac{m\pi}{T}t}$ are orthogonal. Specifically, show that integrating over one full oscillation period of $2T$, the following is true:

$$\frac{1}{2T}\int_{-T}^{T} f(t)g^*(t)dt = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n. \end{cases}$$

(b) Now we are ready to validate the c_n formulation in Eqn 2. Start with Eqn 1, and multiply by both sides $g^*(t)$. Then integrate both sides, averaging over one period of time $2T$ per Eqn 3. Then use the orthogonality relation of complex exponentials to very Eqn 2. It may help to write out several terms of the RHS in Eqn 1. Be careful to note that the sum runs over all integer values of n, but we are $g(t)$ has only one specific value of m.

2. **That's Odd**

Square pulses pop up everywhere in physics and engineering. Such applications include pulses of light being sent down fiber optic cables (intensity vs. time) and digital pulses that underly ever piece of electronics you'll ever use (bluetooth speaker, smartphone, temperature sensor, air quality sensor, etc) The Fourier spectrum is crucial to properly analyzing these. So let's look a little deeper.

Consider an odd square wave function $f(t)$ that discontinuously jumps between a value of 0 and A with a period of $2T$. One period of oscillation is defined by

$$f(t) = \begin{cases} A & \text{for } 0 \leq t < T \\ 0 & \text{for } T \leq t < 2T \end{cases} \tag{4}$$

This pattern just repeats itself over and over.

(a) Sketch the function for time up to $t = 0$ to $6T$

(b) Show that Fourier coefficients for this odd square wave are given by: $a_o = A/2$; $a_n = 0$ for all $n \geq 1$; $b_n = \frac{2A}{n\pi}$ for $n = 1, 3, 5, \ldots$ and $b_n = 0$ when $n = 2, 4, 6, \ldots$.

(c) Make a stem plot of using the real-valued coefficients $\sqrt{a_n^2 + b_n^2}$ vs n. This illustrates the relative strength (intensity) of each harmonic in the series.

(d) Check your work! Flip over to the Fourier Series applet on Falstad: `https://www.falstad.com/fourier/`. Click the 'square' wave and see if your stem plot looks similar!

(e) While here, you can also have some fun synthesizing your own arbitrary waves. Just grab the dots of the stems. You can even play back the tone you constructed. Fun!

3. **String Theory**

Figure 1: Strumming a guitar. Great fodder for solving Fourier series. Image credit: `https://www.uberchord.com/blog/common-beginner-guitar-mistakes/`

Our study of Fourier series was originally motivated by application to music. We talked about how each instrument generates a unique series of harmonics (the ***timbre*** of the instrument). This allows our ears and brains to differentiate a middle-A note played on the piano vs. the guitar vs. the violin vs. cello, etc. When a player strums a guitar string or taps a piano note causing a hammer to strike the string, multiple modes (frequencies) are excited. Exactly which modes are excited and in what proportions depends on exactly how and where the string is plucked or struck.

Now let's go a little deeper into the math. Imagine you have a guitar (or piano or violin) string of length L. In one case, the string is plucked a distance far from the middle $d = L/8$, causing it to displace in the transverse direction a distance $y(L/8) = h$. In the other case, assume the guitar player strums right in the middle of the string at $d = L/2$, such that transverse displacment is $y(L/2) = h$. Thus, the deformation of the string is modeled as follows:

$$y(x) = \begin{cases} \frac{h}{d}\, x & \text{for } 0 \leq x \leq d \\ -\frac{h}{L-d}\,(x-d) + h & \text{for } d \leq x \leq L \end{cases} \tag{5}$$

(a) Plot the transverse displacement $y(x)$ vs. x for the guitar string for both cases ($d = L/2$ and $d = L/8$) for $0 \leq x \leq L$. Carefully mark the values on your plot corresponding for $x = d$ (where the string is plucked/struck) and $y = h$.

(b) Given the waveform $y(x)$ in Eqn 5, set up the integrals need to compute the strenght (intensity) of the nth harmonic of the Fourier series for the guitar string plucked at a distance d from the bridge. Later we'll plug in values of $d = L/8$ and $d = L/2$ to see how musicians can create different "colors" of the same note. For now, Recall the intensity is computed as $\sqrt{a_n^2 + b_n^2} = 2|c_n|$. This means you have a choice of whether you compute the a_n and b_n coefficients or the complex exponential version c_n. One path is likely to offer less resistance than the other, so think carefully before launching into integrals.

(c) Plug in values for $d = L/2$ and $d = L/8$ to find expressions for the Fourier coefficients for these two cases. Again, the point here is to illustrate how the position of the string pluck has a dramatic effect on the Fourier spectrum, and therefore the sound quality the listener perceives. Math meets music, yay! (See more below).

(d) For the guitar (or ukelele) string, make a ***stem plot*** illustrating the intensity of the nth harmonic. Note that matlab has a built-in function `stem()`, that may be helpful.

(e) Based on the stem plots, compare and contrast the harmonic structure of the sound produced by plucking a note in the middle vs. off-center. How might these sound different to our ears? (Ask a musical friend, if you haven't had much training musical training yourself....or try it on an actual guitar string!)

$$c_n = \frac{h}{b^2 d(L-d)} \left[1 - e^{-i2\pi n d/L}\right]$$

where

$$b = -\frac{4\pi^2 n^2}{L^2}$$

4. **My stomach has a heart beat**: As briefly mentioned in class, electrical recordings of stomach electrical waves look sort of like periodic decaying exponentials, defined by

$$v(t) = v_o e^{-t/\tau}$$

These waveforms repeat every T seconds (typically 20 s) for the stomach. The timescale of the waves τ is typically about 2 sec.

Show that the Fourier spectrum goes like (is proportional to):

$$|c_n| \propto \frac{1}{\sqrt{\left(\frac{T}{\tau}\right)^2 + (2\pi n)^2}}$$

Sketch a stem plot or use matlab to plot the relative intensity of each harmonic given by $2|c_n|$. Then explain why Dr. Richards excitement of "Holy cow, you've just discovered multiple pacemakers in the stomach; this will rewrite the medical textbooks on GI electrophysiology forever!" was unfounded. That is, use your results to explain why the electrical recordings don't actually support such a conclusion.