ENGN/PHYS 225 (February 15, 2021)
Complex Numbers: Euler's ID, Waves, and Phasor Math

Complex numbers, trig and Euler's ID

Remember, there is a 3-way equivalence between 1) drawing a triangle; 2) writing a complex number $z = a + jb$, and writing a complex exponential $re^{i\phi}$. Be able to move between any one of these three forms

Adding complex numbers is straight forward, just remember to add the real parts separately from the imaginary parts

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2). \tag{1}$$

Subtraction works similarly.

Exponentials are so nice to use because they have the following properties for multiplying and dividing complex numbers:

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)} \tag{2}$$

Note the amplitudes multiply; the angles add.

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_2-\theta_2)} \tag{3}$$

Note the amplitudes divide; the angles subtract.

Euler's ID is incredibly useful. We'll often use it in this form:

$$re^{i(\omega t+\phi)} = re^{j\phi} e^{i\omega t} \tag{4}$$

where r is the magnitude, $e^i\phi$ expresses the phase angle drawn in the complex plane, and ωt expresses the time dependence (how fast the phasor rotates around the plane in time). Notice the ***time dependence can be written separately from the magnitude and phase information***. This is the mathemagic behind phasors. Also note that the a unit length complex exponential $e^{i\phi}$ simply encodes a rotation (phase angle) in the complex plane. When it multiplies another number, it does NOT change the magnitude.

Complex exponentials are ultimately just a very convenient mathematical tool for expressing sines and cosines. We need to know 3 things to define a wave: 1) amplitude; 2) phase; 3) (angular) frequency. Using Euler's ID we can write oscillating signals as follows:

$$A\cos(\omega t + \phi) = Re[Ae^{i(\omega t+\phi)}] \tag{5}$$

$$B\sin(\omega t + \phi) = Im[Be^{i(\omega t+\phi)}] \tag{6}$$

Some math problems to get (re-)acquainted with complex numbers

1. Sketch each of the following in the complex plane. It may be helpful to write in $z = a + jb$ in complex exponential form $re^{j\phi}$. **Remember, r represents the magnitude and $e^{j\phi}$ represents the phase angle (rotation) relative to positive real axis**!

   (a) $5\,e^{j\pi/2}\,2\,e^{j\pi/4}$

   (b) $\frac{\sqrt{2}+j\sqrt{2}}{j}$

   (c) $\frac{3+j4}{e^{j\pi}}$

   (d) $10\,e^{j\pi/2}e^{j\pi/4}e^{j\pi/4}e^{-j\pi/4}e^{-j\pi/2}\,10\,e^{j\pi/2}$

2. Let's say we have two complex numbers: $z_1 = 3 + 4j$ and $z_2 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$.

   (a) Sketch z_1 and z_2 in the complex plane.

   (b) Write z_1 and z_2 in the complex exponential form.

   (c) Compute and sketch: $z_3 = z_1 + z_2$.

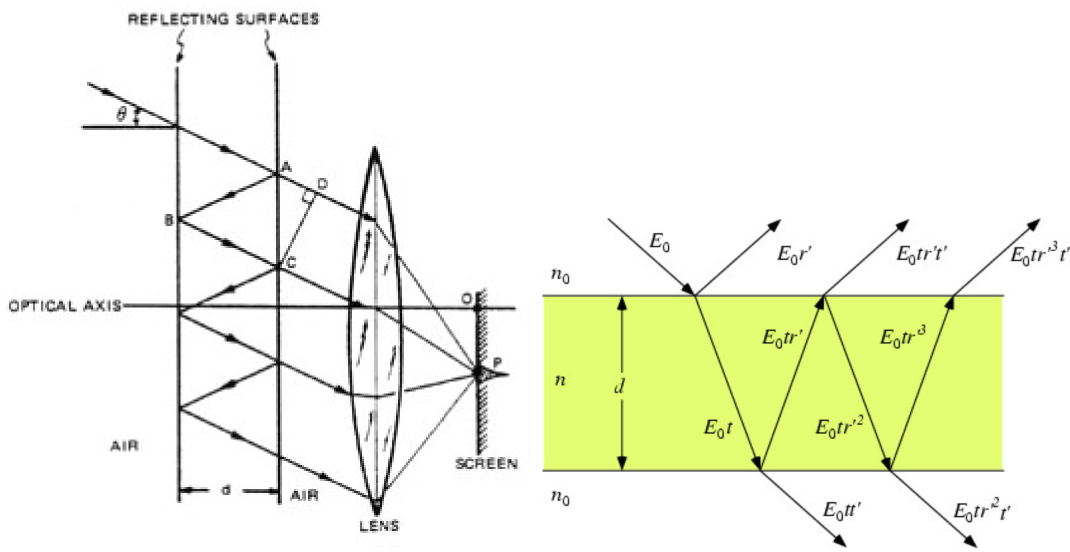
   (d) Compute and sketch: $z_4 = z_1 z_2$. Hint: it may be very helpful to turn each of these into complex exponential form first.

   (e) Compute and sketch: $z_5 = \frac{z_1}{z_2}$.

Application to Optics and Optical Sensing

Time for Reflection The Fabry-Perot interferometer (FPI) is a a widely used instrument in modern optics. Common applications include spectroscopy include environmental monitoring to determine compounds dissolved in water and acidity levels. In materials science, X-ray spectroscopy is used to determine atomic structure of solids. Last but not, a FPI can select a certain modes of a laser—very useful for optoelectronic communications and biomedical sensors.

The FPI working principle is interference (superposition) of many waves to produces narrow-bandwidth output. Waves constructively interfere only if they have just the right phase delay; otherwise, all of the superposing waves tend to cancel each other out. In other words, if white light goes in, only a very specific narrow band of wavelengths (colors) is transmitted to the other side—hello, lasers! This interference through a summation of many waves can be produced either by passing incident light through a precisely machined glass block, or by two mirrors, as illustrated in Fig. 1. Now let's get down to some math!



(a) Fabry Perot etalon

(b) Superposition of transmitted waves

Figure 1: Fabry-Perot Interferometer wave interference/superposition. Image credits: (a)`http://pe2bz.philpem.me.uk/Lights/-%20Laser/Info-902-LaserCourse/c10-05/Module10-5.htm` and (b)`http://scienceworld.wolfram.com/physics/Fabry-PerotInterferometer.html`

1. Draw a phasor diagram to represent each transmitted component when many waves destructively interfere. Using a graphical argument, show that the magnitude of the output wave is very small

2. In class, we discussed that the transmitted electromagnetic wave can be represented by:

$$\begin{aligned} E_{transmitted} &= E_1 + E_2 + E_3 + \cdots && (7) \\ &= E_o tt' \sin(\omega\tau + \delta) + E_o tt' r^2 \sin(\omega\tau + 3\delta) + E_o tt' r^4 \sin(\omega\tau + 5\delta) + \cdots && (8) \\ & && (9) \end{aligned}$$

In the Eqn. 7, the constants t and $t^{'}$ and transmission coefficients and r is a reflection coefficient; they are all < 1; τ represents time (those sneaky opticians already claimed t and $t^{'}$ for transmission coefficients in this problem!), and $\delta = 2\pi d/(\lambda/n)$ is the phase delay incurred due to the extra path length traveled. Here, d and n represent, respectively, the thickness and index of the of the glass plate within the etalon.

Now let's do some math! Show that for the transmitted wave can be represented using with complex exponentials as:

$$\frac{E_{transmitted}}{E_o} = Im\left[\frac{tt^{'}}{1 - r^2 e^{i2\delta}} e^{i(\omega\tau + \delta)}\right] \tag{10}$$

Hint: There's a geometric series being summed here with a ratio of $r < 1$.

3. Now let's take it a step further. The transmission intensity is defined as

$$\left|\frac{E_{transmitted}}{E_o}\right|^2.$$

Given the definitions of $R = |r^2|$ and $tt^{'} = 1 - r^2$, show that

$$\left|\frac{E_{transmitted}}{E_o}\right|^2 = \left|\frac{tt^{'}}{1 - r^2 e^{i2\delta}}\right|^2 = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2\delta}. \tag{11}$$

Friendly reminder/hint: This problem was formulated as sines not cosines, so which part, real or imaginary do you need to consider to get from the complex plane back to physical reality?

Batman goes to the monster truck rally.

Figure 2: Batmobile is now converted for monster truck rally. Image credit: denver.cbslocal.com.

Whoa, check out the Batmobile (2), recently converted for the monster truck rally. Pretty sweet setup, eh? Even the Joker and Riddler are jealous. While at the starting line, Batman is revving his engine, which sits atop a chassis ludicrously large struts to absorb sudden impacts and dampen the mechanical vibrations. While Batman and friends were designing the truck, they got really interested in physical models and mathematical solutions for how the chassis would perform while cruising down the highway to get to the rally. They drew up a nice little model on the back of an envelope. The parallel spring k and damper c represents the struts—they are springy and they dampen vibrations. The mass m is batman and everything else sitting atop the suspension system that vibrates. Batman and friends represents the force imparted from massive V8 engine (not the tomato juice) to the chassis as $F(t) = F_o \cos \omega t$. They know that the reaction spring force is given by $F_s = kx(t)$ and the force due to damping is given by $F_c = c\frac{dx}{dt} = c\dot{x}$. They want to solve for the steady-state oscillations $x(t)$ but haven't taken math methods before. So, they want—check that, need!— your help.

1. Given the diagram in figure 3, draw a free body diagram and use Newton's Law to show that

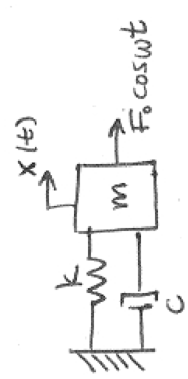


Figure 3: Harmonic excitation of spring-mass-damper system. This model is ubiquitous in many mechanical problems; in this case it represents vibrations of an automobile caused by the engine.

the position of the mass vs. time, $x(t)$ obeys the equation[1]:

$$m\ddot{x} + c\dot{x} + kx = F_o \cos\omega t \tag{12}$$

2. As we've done before in class, we can often make smart guesses for solutions to ODEs based on physical intuition alone. In this case, let's guess that

$$x(t) = x_o \cos(\omega t + \phi_x)$$

solves the 2nd ordinary differential equation (ODE) in Eqn 12. Here, x_o quantifies the ***magnitude*** (modulus) of oscillations; the phase angle ϕ_x describes the ***phase*** of the displacement relative to the forcing.

Now let's move from the time domain into the complex plane, where we can temporarily ditch the time-dependence to make it easier to solve for the magnitude and phase. Show that we can then write:

$$\underbrace{(-\omega^2 m + j\omega c + k)}_{\tilde{Q}}\underbrace{x_o e^{j\phi_x}}_{\tilde{X}} = \underbrace{F_o}_{\tilde{F}} \tag{13}$$

3. In Eqn 13, we have defined the complex quantities $\tilde{Q}$, $\tilde{X}$, $\tilde{F}$. For mathematical convenience, write $\tilde{Q}$ in the form $Ae^{j\phi_q}$, and define A and ϕ_q in terms of ω, m, k, and c.

4. Not so bad, so far, eh? Complex numbers are about to make our lives really easy! Check this out. We have an equation that relates three complex numbers together: $\tilde{Q}\tilde{X} = \tilde{F}$. Therefore, show the following are true regarding the **magnitude** of the vibrational response:

$$x_o = \frac{Fo}{\sqrt{(k-\omega^2 m)^2 + (\omega c)^2}} \tag{14}$$

We can rewrite Equation 14 as follows (Eqn 15)

$$\frac{x_o}{F_o/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{15}$$

where we define:

   (a) $\omega_o = \sqrt{k/m}$. This is the "***natural frequency***" of the system, the frequency (rad/s) at which it wants to oscillate.
   (b) $r = \omega/\omega_o$. This is the dimensionless ratio of the forcing frequency relative to the natural frequency
   (c) $\zeta = \frac{c}{2m\omega_o}$. This is the damping ratio. Higher ζ values mean more damping.

5. Also, show that the following expression quantifies the ***phase*** or ("phase shift") between the forcing $F(t)$ and the displacement $x(t)$:

$$\phi_x = -\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \tag{16}$$

[1]There's a subtle point here that an mg term does not appear. In vibrational analysis, we typically assume we are measuring around the static equilibrium point δx_o. In this case, the spring will be "pre-compressed" such that the $mg = k\delta x_o$. These terms continue to be equal and opposite, thus canceling out

6. To wrap things up, Batman is cruising down I-81 on his way to the next big monster truck rally. The engine is slowly churning at 1600 rpm (which sets ω in this problem), and generates a vertical force magnitude of $F_o = 2000$ N. The mass undergoing oscillation is $m = 500$ kg. The struts installed this version of a batmobile are known to have a spring constant of $k = 10^5$ N/m and damping constant of $c = 2000$ Ns/m. Sketch the ***phasors*** representing $\tilde{F}$ and $\tilde{X}$. In addition, carefully plot $F(t)$ and $x(t)$ versus time, showing about 5 total cycles of oscillation cycles.

Batman thanks you for this excellent analysis, and speeds away with a parting "Peace, and keep it real!".