Problem Set 1
ENGN/PHYS 225—Winter 2021
Due Date: Friday, 05 Feb, 4pm

Foreword

This assignment plays with gradient, divergence, and curl. Remember, visualization leads to intuition leads to really powerful ways of thinking about and solving math problems. So strain that side of your math brain the best you can.

Problems

1. **Surf's Up!**

Figure 1: Kerby Brown surfing the fearsome break at Teahupoo Tahiti. Image credit: `https://www.redbull.com/ca-en/sessions-teahupoo-tahiti-october-2018`.

At a gnarly surf break, the swell is pumping, creating perfect waves. The height of the wave as a function of position (x, y), where x is the north-south direction, and y represents the east-west direction is given by:

$$h(x, y) = e^{x/a} + bxy^2$$

where a and b are constants describing the "peakiness" of the wave. Let's take $a = 5$ and $b = 1/2$. Furthermore, to make sure the height of the wave doesn't shoot off to infinity, we could restrict the domain of $h(x, y)$ to something sensible values (such as $x \in [-10, 10]$ and $y \in [-20, 20]$).

(a) Sketch or make a computer generated plot of the wave $h(x, y)$.

(b) Fearless Fred is bombing down this wave, contemplating his next move. Compute the directional derivative at $(x, y) = (4, 6)$ in the following directions: (-1, 0), (0, -1), and (-1, -6).

(c) Interpret what you just computed: how much will Fred's elevation change when taking a unit step in each of these directions?

(d) In what direction should Fred point his board in order to descend the wave as fast as possible (given he's currently at $(x, y) = (4, 6)$)?

(e) In what direction should Fred point his board in order to momentarily cruise at a constant elevation given he is currently at (4,6)?

2. **Making the Transition: Boundary Layers**

Boundary layers of substantial interest in fluid mechanics with application to aeronautical, chemical, nuclear, and biomedical engineering. First time hearing the term boundary layer? Fear not! As defined in Coulson and Richardson's chemical engineering text:

> *When a fluid flows over a surface, the part of the stream that is close to the surface suffers a significant retardation, and a velocity profile is developed in the fluid. The velocity gradients are higher at the surface and progressively smaller with distance from the surface...The flow conditions in the boundary layer are of considerable interest to chemical engineers because these influence not only the drag effect of the fluid on the surface but also the heat or mass transfer rates where a temperature or a concentration gradient exists.*

At this point you may be asking "That's cute, but what has this got to do with math methods and/or real life?" Glad you asked!

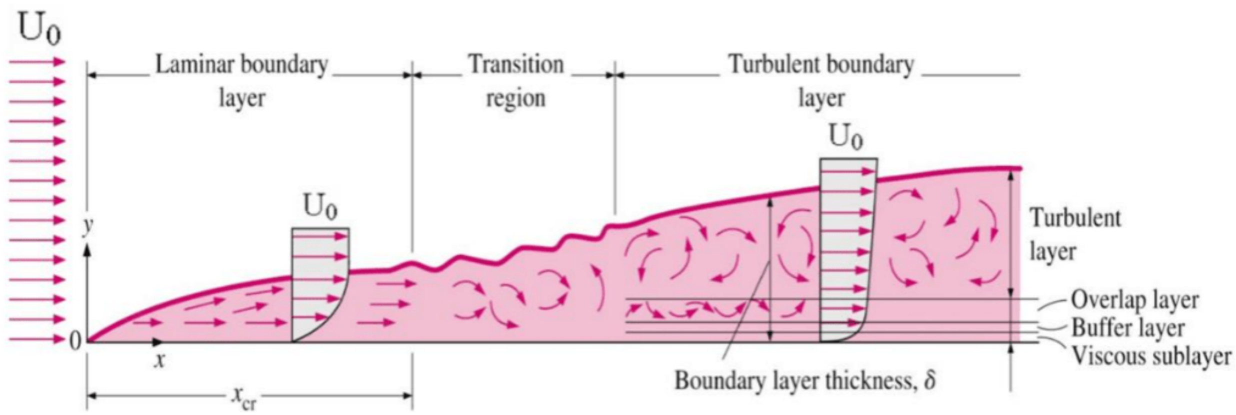

Figure 2: Boundary Layer Transition from Laminar to Turbulent Flow. Image credit: `https://www.nuclear-power.net/nuclear-engineering/fluid-dynamics/boundary-layer/`

Boundary layers are illustrated in Figure 2. Note the flow starts nice and laminar ("orderly flow") then transitions to turbulent ("all mixed up") further downstream. Notice the little swirls (known as "eddies"). Swirls....math...could it be ***the curl***?!

Let's dive deeper into this fluid mechanics problem to see how and why the curl is important[1].

Firstly, the real world stuff: Note the transition zone. Want more efficient airplanes? Get rid of turbulence (good luck!) or at least try to minimize it. Want chemical mixing to occur? Then maybe you want turbulence to stir things up. In biological fluid flow, laminar is generally good, turbulence is generally bad.

Secondly, the mathy stuff. Figure 3 illustrates the velocity profile of a boundayr layer: the velocity along a plate (airfoil, pipe wall) is zero at the boundaryowing to the "no slip condition". The flow velocity increases traveling away from the the plate until it reaches a constant ***freestream velocity*** U_o.

[1]By no means is the following a full treatment of all of the fluid mechanics involved—we're neglecting Reynold's number, pressure gradients and fluid viscosity herein—but hopefully will help you build some intuition. Take Engn 311/351 to learn more!

We can express this velocity field as:

$$\vec{u}(y) = U_o \left[1 - e^{-y/d}\right] \hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}$$

where U_o is the freestream velocity (constant value), and d defines the boundary layer thickness. Note that $\vec{u}$ varies only as a function of y in this problem.

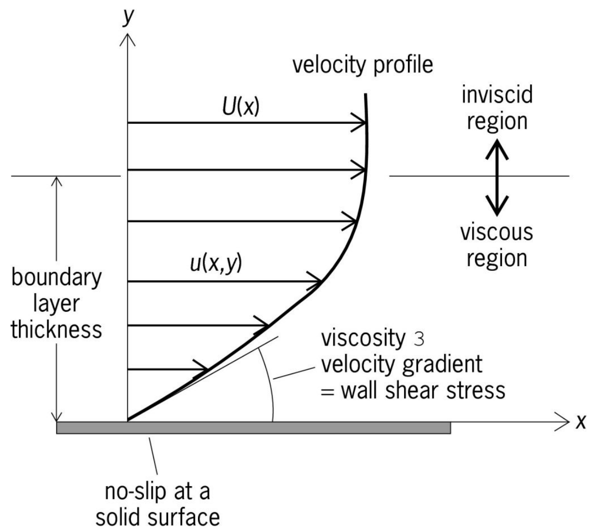

Figure 3: Velocity profile of boundary layer. Image credit:`https://encyclopedia2.thefreedictionary.com/boundary-layer+flow`

(a) What would happen to microscopic paddle wheels pinned in the flow: would they be made to rotate? If so, in which direction (CW or CCW)? Would a paddle wheel spin faster if it placed to closer to or further from the wall? Employ intuitive physical reasoning as necessary.

(b) Develop an expression for the ***curl*** of the velocity field $\nabla \times \vec{u}$. Provide an intuitive interpretation of this math expression. It may help to reference your answer to part a.

(c) Let's say the freestream velocity is $U_o = 10$ m/s and $d = 10$ cm $= 0.1$ m. Compute and compare the rate of rotation of a little paddle wheel placed at the $y = 0, 5, 10, 50$ cm. Do your results agree with your intuitive answer from part a?

(d) Relate your math answers above back to Figure 2. That is, how does your math result explain (at least in part) how the swirly twirly eddies in the turbulent region develop?

(e) Lastly compute the ***divergence*** of the velocity field $\nabla \cdot \vec{u}$. Note this isn't a random math exercise. In fluids, an ***incompressible fluid*** has zero divergence [2] Not all fluids are incompressible, by the way.

[2] This derives from the continuity equation $\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \vec{u})$. If fluid density ρ is constant, it implies the divergence of the velocity field $\nabla \cdot \vec{u} = 0$.

3. **Wunderbar!**

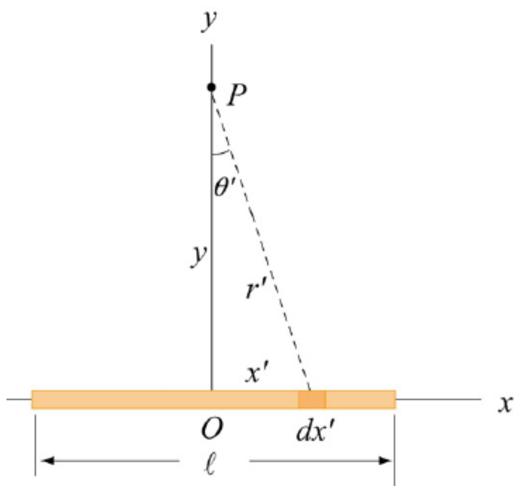

Figure 4: Finite length bar, all charged up. We're working on the y axis in this problem. Image credit: `http://web.mit.edu/viz/EM/visualizations/coursenotes/modules/guide03.pdf`

Imagine you have a finite length bar of length l and linear charge density $\lambda = q_{total}/l$. Perhaps this could be the charge stored in a small component of an an electrical circuit, or perhaps a segment of an electrically active of muscle tissue, etc. The bar is centered so that the y-axis runs perpendicular to its center point. Now let's consider the electrical potential as a function as a function of y. A little bit of E&M derivation will reveal that the electric potential is given by:

$$V(y) = \frac{\lambda}{4\pi\epsilon_o} \ln\left[\frac{1+\sqrt{1+(2y/l)^2}}{-1+\sqrt{1+(2y/l)^2}}\right]$$

(a) Let's consider that we are very close to the bar such that $y/l << 1$. In this case, the expression for the eletric potential simplifies to[3]:

$$V(y) \approx \frac{\lambda}{2\pi\epsilon_o} \ln(l/y)$$

(b) Assuming we are close to the charged bar, per part a, find an expression for the electric field component in the y-direction E_y (the component that points in the direction of $\hat{j}$). Recall that the electric field is the negative gradient of the electric potential $\vec{E} = -\vec{\nabla}V$. Note: this equation encodes the fact that positive charges flow down the gradient from higher to lower potential energy, just like a marble rolls down a hill due to the gravitational field. In any event, your answer should be a function of y and other physical constants.

(c) Make a **hand sketch** of the result for E_y.

(d) Find an expression for the divergence of the y component of the electric field. Physically, what does this tell you about E_y? Is it what you'd expect on intuitive grounds?

[3]Wondering where this bit of mathemagic came from? As the classic exercise to the reader, employ the binomial approximation $(1+x)^n \approx 1+nx$ for $x << 1$.

4. **Dealer's Choice** This dealer's choice is intended to let you choose a problem where you feel you have a mathematical weak spot for the gradient, divergence, or curl. Pick (at least) one of these concepts, then do one of the following:

   (a) Make up your own problem—hopefully in an application you feel is fun and relevant—write the problem statement and the solution for it.

   (b) Not feeling particularly inspired? No prob. In which case, look up a problem in the textbook or other legit resource, and work through a solution for it.

   In any event, clearly state the problem at hand and the solution method. Lastly, include a brief (2-4 sentence) reflection about how this problem helped you emerge with a better understanding of a concept you originally found challenging. As well, reflect upon any concepts or techniques where you still feel shaky. (We'll work together to iron out the kinks!).