The Big Ω: What the heck is digital frequency anyway?

This tech note concerns the definition and usage of the digital frequency Ω and compares/contrasts to the angular frequency ω.

A signal $x(t)$ representing some physical quantity (river height vs. time, electrode voltage vs. time, strain due to gravity waves, whatever) is sampled at regular intervals spaced T apart. We call T the ***sampling period***. Thus, the corresponding ***sampling frequency*** is given by $f_s = 1/T$. The discrete times at which we sample the signal are given by

$$t_n = nT = \frac{n}{f_s} \qquad \text{where} \qquad n = 0, 1, 2, \ldots$$

So we can think of n as the integer index keeping track of time.

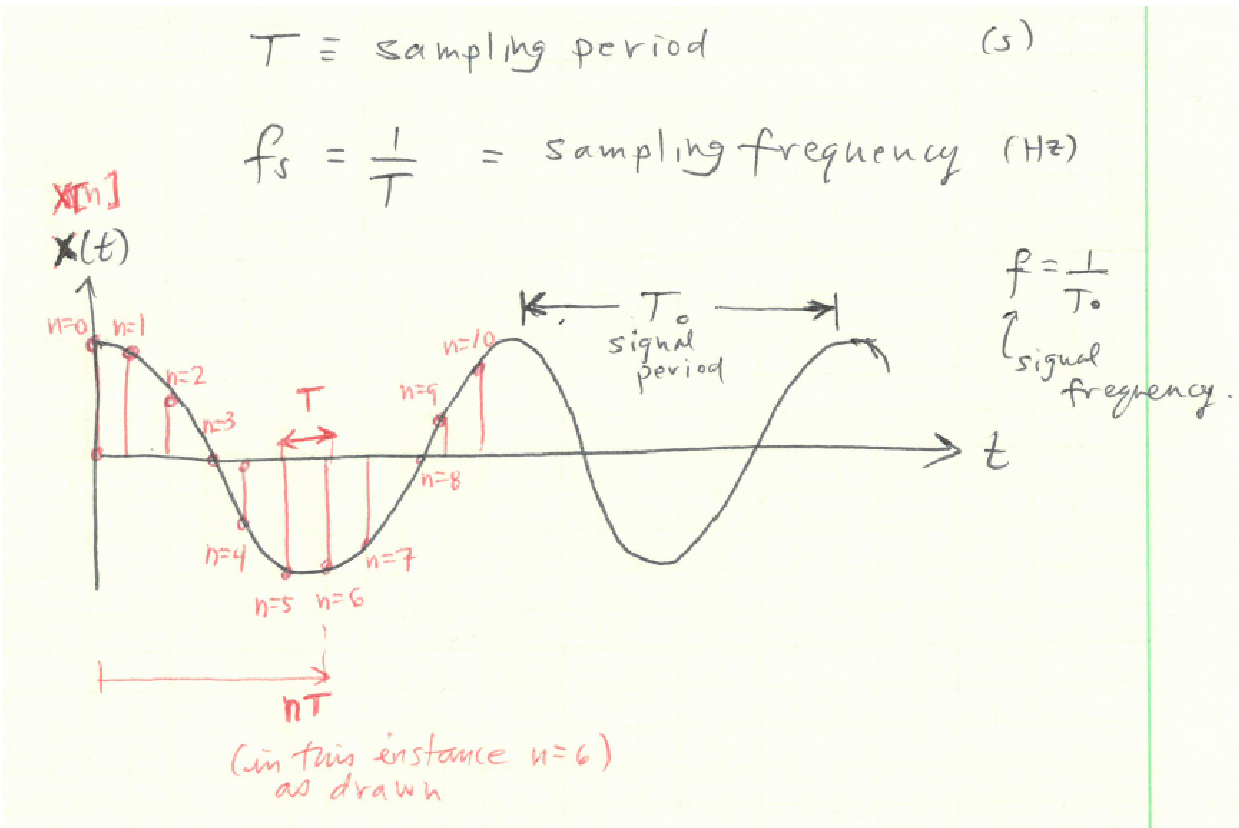

Figure 1: An oscillating signal $x(t)$ bobs up and down with frequency f_o signal period T_o. The digital sequence $x[n]$ is sampled at discrete times $t_n = nT$, where T is the sampling period.

Now if our underlying signal is given by:

$$x(t) = A\cos(\omega t + \phi)$$

and we sample it at discrete times t_n, our digital sequence is given by:

$$\begin{aligned}
x[n] &= A\cos(\omega t_n + \phi) && (1)\\
&= A\cos(\omega(nT) + \phi) && (2)\\
&= A\cos((\omega T)n + \phi) && (3)\\
x[n] &= A\cos(\Omega n + \phi) && (4)\\
& && (5)
\end{aligned}$$

Here we have defined the **digital frequency** $\Omega = \omega T$. Note that our digital sequence sequence $x[n]$ still traces out an oscillating signal as integer time index n advances.

Note also that the digital frequency Ω is the angular frequency ω (rad/s) times the sampling period T (s/sample), so the units digital frequency are rad/sample. One nice interpretation of Ω therefore, is how many radians the underlying oscillation progressed through from one digital sample to the next. For example, if $\Omega = \pi/10$ rad/sample, the oscillation has advanced only a "little bit", namely $\pi/10/(2\pi) = 1/20$ of one full oscillation. By contrast if $\Omega = \pi$ rad/sample, then the physical oscillation has already undergone one-half of a full oscillation. We'll see later that this latter case has profound consequences in signal processing land.

One major reason for using Ω instead of ω is that signal processing techniques fundamentally take in a sequence of numbers $x[n]$ and operate on them in the same way regardless of the sampling period. By generalizing sequences to think of time as an integer index, we can generalize signal processing techniques. For example, our river rock data may be sampled at $f_s = 40$ Hz, while our ECG data was sampled at $f_s = 360$ Hz. If we want to apply, say, a moving mean filter, the math will work out more nicely if we use Ω instead of ω.

In essence, we are simply normalizing the frequency of the physical signal to how fast it was sampled. Specifically:

$$\begin{aligned}
\Omega &= \omega T && (6)\\
&= \frac{\omega}{fs} && (7)\\
&= \frac{2\pi f}{fs} && (8)\\
\Omega &= 2\pi \frac{f}{fs} && (9)\\
& && (10)
\end{aligned}$$

So the digital frequency is just 2π times the frequency of the actual physical signal f normalized to the sampling rate f_s.