# Discrete Fourier Transform (DFT): What frequency components are present in the signal?

This tech note concerns some core ideas and some math details of the DFT and well as some very practical considerations. Keep in mind the goal of the DFT is this: It decomposes a sampled signal $x[n]$ into a set of DFT coefficients $X(m)$, which tell which frequency components are present in the signal as well as their relative intensity.

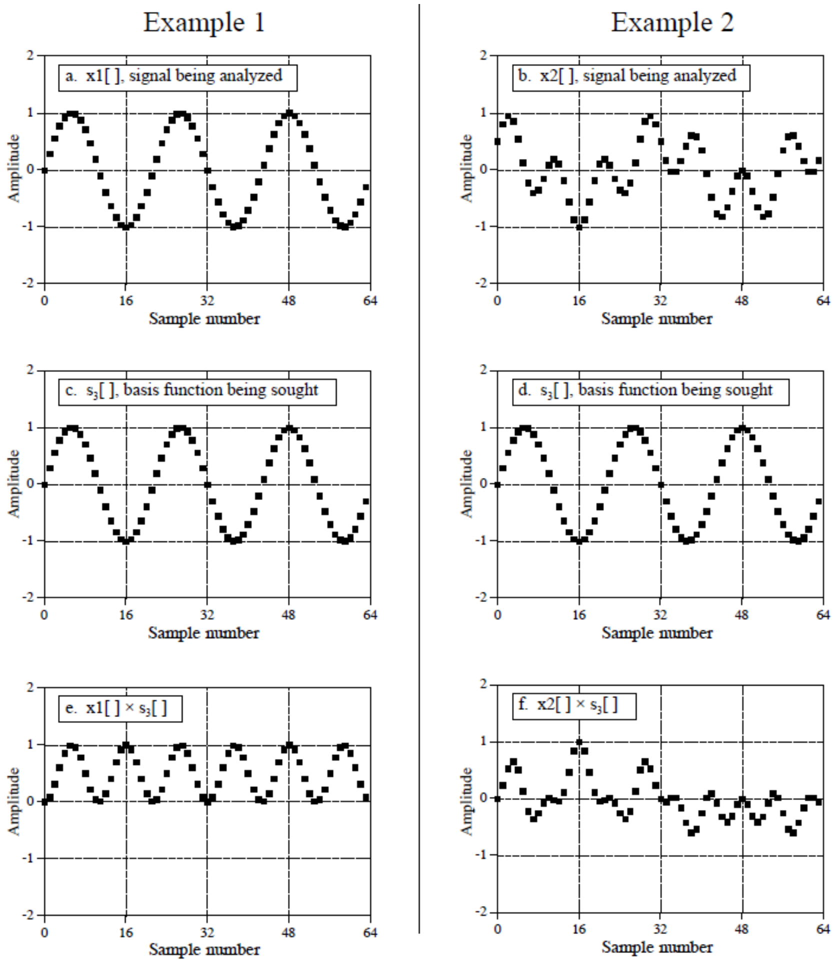

Figure 1: Illustrating the core idea of the DFT as the correlation between the sampled sequence and the basis function oscillating at a frequency $m\omega_o$. The DFT especially multiplies these two together element-wise, then adds up the result via the sum (can think of this similar to area under the curve. From Steven Smith's "The Scientist and Engineer's Guide to Digital Signal Processing" `http://www.dspguide.com/ch8/6.htm`. (Image used without permission)

A signal $x(t)$ representing some physical quantity (river height vs. time, electrode voltage vs. time, strain due to gravity waves, whatever) is sampled at regular intervals spaced $T$ apart. We call $T$ the ***sampling period***. Thus, the corresponding ***sampling frequency*** is given by $f_s = 1/T$. The discrete times at which we sample the signal are given by

$$t_n = nT = \frac{n}{f_s} \qquad \text{where} \qquad n = 0, 1, 2, \ldots, N-1.$$

So we can think of $n$ as the integer index keeping track of time and we write the digitally sampled N-point signal as $x[n]$.

## 1 DFT is just correlation of digital signal $x[n]$ with sine wave oscillating at frequency $\omega_m$

The fundamental question we would like to address: What are the underlying frequency components in $x[n]$. The DFT tell us the answer! All the DFT really computes is a ***correlation*** between the sampled signal $x[n]$ and a basis function oscillating at an angular frequency of $\omega_m$ where:

$$\omega_m = m\omega_o = m\frac{2\pi}{NT}$$

where we identify the ***fundamental frequency*** $\omega_o = \frac{2\pi}{NT}$. Note that $NT$ is the total duration for which the signal was sampled. Thus, the fundamental frequency is a (co-)sine wave that completes exactly one oscillation over the recording duration. Furthermore a signal oscillating at $\omega_m$ will oscillate $m$ times total over the recording duration.

In specific math terms, we are to correlate our sampled signal $x(t_n) = x[n]$ with a family of (co-)sine functions—aka our ***DFT basis functions***—oscillating at a frequency of $\omega_m$. Say whaaaaaa? Just take a look at Figure 1. Once you understand this graphic, you have the keys to the DFT Kingdom.

The basis functions $s_m(t_n) = s_m[n]$ can be expressed in complex exponential form via Euler's Formula: $e^{j\omega_m t_n} = \cos(\omega_m t_n) + j\sin(\omega_m t_n)$.

$$\begin{aligned} s_m[n] &= e^{-j\omega_m t_n} && (1)\\ &= e^{-j(m\omega_o)t_n} && (2)\\ &= e^{-jm(\frac{2\pi}{NT})t_n} && (3)\\ &= e^{-jm(\frac{2\pi}{NT})(nT)} && (4)\\ &= e^{-jm(\frac{2\pi}{N})n} && (5)\\ &= e^{-j2\pi mn/N} && (6)\\ & && (7)\end{aligned}$$

So the net sum (literally!) is $X[m]$, a complex number whose magnitude $|X[m]|$represents the intensity with which $x[n]$ oscillates at an (angular) frequency of $\omega_m = m\omega_o$:

$$\begin{aligned} X[m] &= \sum_{n=0}^{N-1} x[n]s_m[n] && (8)\\ &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi mn/N} && (9)\end{aligned}$$

Note the sum runs over the integer time-index $n$ thereby summing over all points in the data series. Again, the index $m$ indexes the frequency of oscillation. We need to compute the sum over a range of values for $m = 0, 1, 2, \ldots, N-1$. Thus we will get a series of coefficients $\{X[0], X[1], X[2], \ldots X[N-1]\}$. We call this series of coefficients $X(m)$ the DFT of the time series $x[n]$.

Why should there be only $N-1$ terms in the DFT? Well, let's answer that question with another question: What happens with $m = N$? How does that compare to when $m = 0$? Hold onto this thought for a bit, we'll address it in the next section.

For now, we can visually represent the information contained in the DFT as a***frequency spectrum***, like the one plotted in Figure 3. This is actually a plot of $|X[m]|/N$ vs. $f = 2 * pi * \omega_m$. For this particular data set, we know that we have $N = 51001$ data points, sampled at intervals of $T = 0.01$s ($f_s = 100$ Hz). Thus, the fundamental frequency, in this case, is $\omega_o = \frac{2\pi}{NT} \approx 0.0126$ rad/s. The dominant peak occurs at a frequency of $f \approx 0.91$ Hz or $\omega = 2\pi f \approx 5.72$ rad/s. This corresponds to the 455th term in the series: $m = 455 \approx \frac{5.72 \text{ rad/s}}{0.0126 \text{ rad/s}}$.

# 2 Important Properties of the DFT

## 2.1 DFT is periodic with period $N$ (number of samples in data series $x[n]$)

Show that $X(m \pm kN) = X(m)$ for any integer value of $k$.

Hint: Just use the property that $e^{ab} = e^a e^b$. Furthermore $e^{2\pi k} = 1$ for any integer $k$.

What's the practical impliciatino? Let's say you want to compute the DFT at "really high" 'frequencies. This is like saying you want the DFT at really high values of $m$, e.g. large values of $\omega_m$. Well, you won't gain any new information by computing at any value of $m > N-1$, because the DFT series $X(m)$ just repeats itself.

So it seems we are fundamentlly limited to computing the DFT up to frequencies of $\omega_m = \omega_{N-1} = (N-1)\omega_o$.

But wait, it gets better (or worse?)...

## 2.2 Frequency folding (or mirroring)

It turns out the DFT also is symmetric about the $m = N/2$. Thus, the DFT is mirrored about the frequency $\omega_m = \omega_{N/2}$.

In specific math terms:

$$|X(N-m)| = |X^*(m)|$$

where * denotes complex conjugate. This is easy to show the general case Note that at $m = N/2$ we have that $|X(N - N/2)| = |X^*(N/2)|$, which is a true story so long as $x[n]$ is real valued. Of course it is in this class! The data series $x[n]$ represents physical quantities: river height, ECG voltage, acceleration, mechanical strain, etc.

So we'll expect to see DFTs mirrored around the frequency $\omega_{N/2} = \frac{N}{2}\omega_o = \omega_{max}$. Whoa, cool and crazy result!

Note that we can recast this max DFT analysis frequency in terms of the sampling rate $f_s = 1/T$ Iunits of Hz) as follows:

$$\begin{aligned} f_{max} &= \frac{\omega_{max}}{2\pi} && (10) \\ &= \frac{1}{2\pi}\left(\frac{N}{2}\frac{2\pi}{NT}\right) && (11) \\ &= \frac{1}{2\pi}\pi/T && (12) \\ &= \frac{1}{2\pi}\pi f_s && (13) \\ f_{max} &= f_s/2 && (14) \end{aligned}$$

Whoa, another cool and crazy result! In practical terms this is the highest frequency we can analyze is one-half of the sampling rate. For instance, if we sample data at $f_s$ 100 Hz, the highest frequency content we can analyze is $f_{max} = 50$ Hz.

Intuitively, this should make sense, we need to get 2 samples for every oscillation in order to see it clearly—one at the crest, one at the trough. This result is so important it is given a special name: The ***Shannon/Nyquist sampling criterion***. What we called $f_{max}$ above is typically referred to as the ***Nyquist rate***. (Henry Nyquist was a brilliant engineer at Bell Labs around the 1930s. You also have him to thank for phasors.)

You'll note that in Matlab, we often plot the "single sided spectrum". See ¿¿ doc fft or online page for matlab fft() here: `https://www.mathworks.com/help/matlab/ref/fft.html`.

Finally, you need to know about ***aliasing***. Due to the mirroring of the DFT around $f_s/2$ as described above, a signal component actually oscillating at a frequency of $f_s/2 + \Delta f$ appears in the DFT at the aliased frequency $f_s/2 - \Delta f$. For example, say we are sampling an ECG signal at $f_s = 100$ Hz. Say there is some strong signal at 60 Hz (the frequency of the US power grid!). This is $\Delta f = 10$ Hz above $f_s/2 = 50$ Hz. So the he DFT will show a strong peak at 40 Hz. Of course, the real artists was the 60 Hz, but now shows up under a new name, an alias, of 40 Hz. These are indistinguishable from one another—which is generally bad news because it confounds signal analysis.

## 3 Real life example using Matlab

Lastly, let's see what all of this is good for in practice! Check out Figures 2 and 3. The former plots accelerometer and gyro data obtained during a human walking gate. Note there are about 1 steps/s. Now check out the corresponding DFT (actually computed using the FFT algorithm).

Note the large peak at about 0.9 Hz, that is 0.9 steps per second, which agrees well with our visual analysis of the time-domain signals. Note the square wave-ish nature of the waveform for acceleration in the $x$-direction which produces the harmonics at integer multiples of 0.9 Hz. Also note: this is single sided spectrum. That's why the frequency axis ends at 50 Hz $= f_s/2$ (this data set was recorded at 100 Hz sampling rate). These were produced following matlab's fft() example.

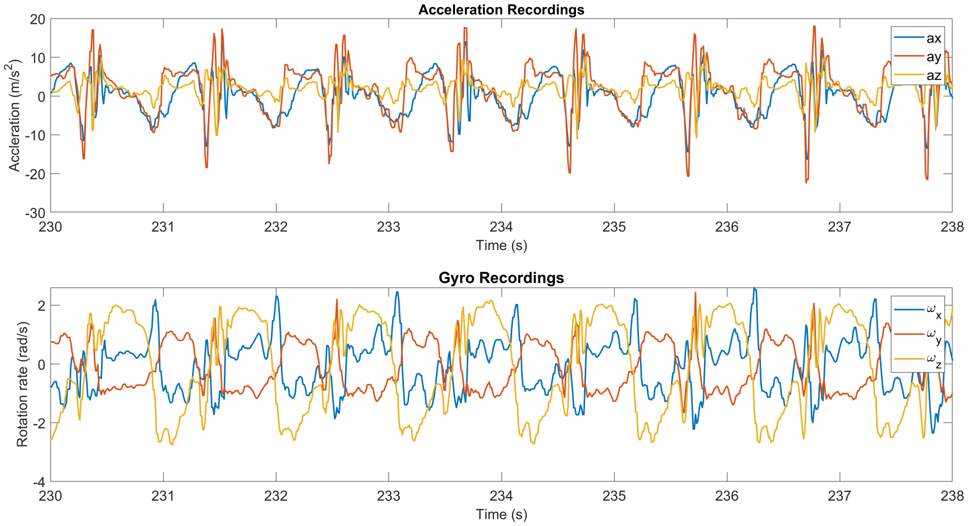

Figure 2: Accelerometer and gyro signals during walking gait. Data from Vajdi et al: `https://arxiv.org/abs/1905.03109`. Note the cylic nature of the signal at about 1 Hz, or about 1 step per second.

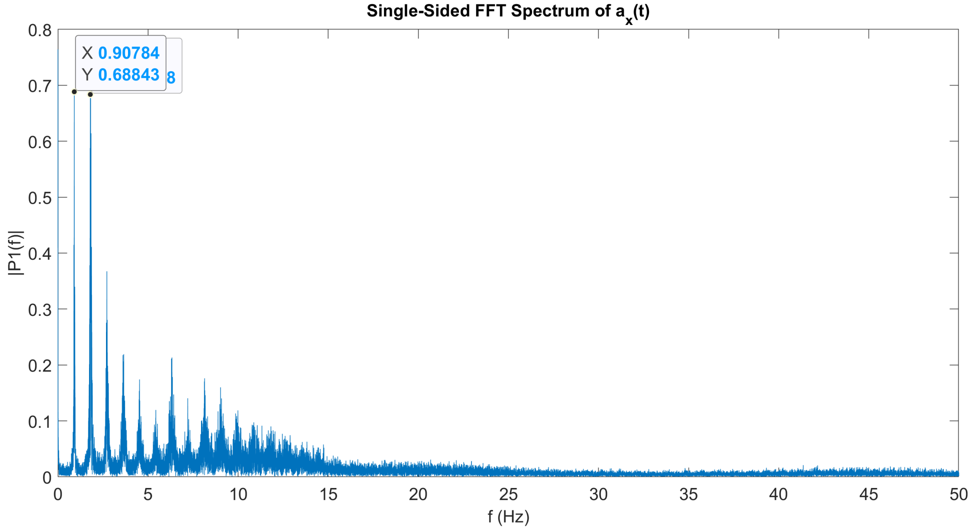

Figure 3: Fast Fourier Transform of acceleration in x-direction. The dominant peak occurs at about 0.9 Hz. Almost all frequency content is contained at frequencies less than 15 Hz.