Z-best: Some problems using the z-transform to build intuition (Project 05)

1. ***Get a move-on: the moving mean filter***

   Consider the classic moving mean filter of length N. For this case, we have filter coefficients $b[n] = 1/N$ for $n = 0, 1, \ldots, N-1$.

   We previously showed that the Z-transform is given by (see `https://erickson.academic.wlu.edu/files/courses2020/sigproc_s2020/readings/Ztransform.pdf`):

$$B(z) = \frac{1}{N}\left(\frac{1-z^{-N}}{1-z^{-1}}\right) \quad (1)$$

$$B(e^{j\omega_m T}) = \frac{1}{N}\left(\frac{1-e^{-j\omega_m TN}}{1-e^{-j\omega_m T}}\right) \quad (2)$$

   Let's say we have a window length of $N = 10$. And let's say we have a sampling frequency of $f_s = 100$ Hz, so the sampling period of $T = 1/100$ s. (For instance, we might be making pressure readings with SmartRock 100 times per second and want to take a running average, aka the moving mean.)

   (a) Over how much time does this filter operate? Therefore, what is the equivalent cutoff frequency for this moving mean filter?

   (b) Sketch the resulting vectors being summed in the complex plane for the case $m = 0$. Also indicate the sum of these vectors. This sum indicates the response of a filter to a dc component (constant, non-oscillating component).

   (c) Now do the same, but the case $m = 4$. This is the response of a filter to at a higher frequency oscillation, specifically a digital frequency of $\omega_4 = 4\omega_o$.

   (d) We previously argued that the magnitude of the Z-transform $B(z)$ of $b[n]$ can be written as follows:

$$\left|B(e^{j\omega_m T})\right| = \frac{1}{N}\frac{\sin\left(N\omega_m T/2\right)}{\sin\left(\omega_m T/2\right)} \quad (3)$$

   where in this case $\omega_m = m\frac{2\pi}{NT}$, for an N sample sequence, so Eqn 3 simplifies further:

$$\left|B(e^{j\omega_m T})\right| = \frac{1}{N}\frac{\sin\left(m\pi\right)}{\sin\left(\frac{m\pi}{N}\right)} \quad (4)$$

   Starting with the $B(z)$, derive the relation in Eqn 4. Then, sketch the magnitude response of this filter in the frequency domain. To help get you started:

   Hints: Think about trig IDs; remember that the magnitude of a complex number is $|z| = \sqrt{z^* z}$, where $z*$ is the complex conjugate of z; and you can possibly find useful the identity: $sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j}$.

2. ***Bread and butter: the Butterworth filter***

   Let's say we have a digital signal sampled at $f_s = 200$ Hz. For instance, this digital signal might be samples from an accelerometer on a bridge to monitor its structural health.

   Let's say we want to filter this signal to rid of various noise components, so we want to construct the following digital filters of the Butterworth variety with a cutoff-frequency of $f_o = 10$ Hz. Note that we have a digital cutoff frequency of $\Omega_o = 2\pi \frac{f_o}{f_s} = 2\pi \frac{10}{200} = \frac{\pi}{10}$ rad/sample.

   (a) Low-pass filter, 1st order
   (b) Low-pass filter, 2nd order
   (c) High-pass filter, 1nd order
   (d) High-pass filter, 2nd order

   For each of these four cases:

   (a) Use Matlab to find the Butterworth filter coefficients b and a.
   (b) Plot vectors in the complex plane representing the terms in the sum of the Z-transform for a digital frequency of $\Omega = 0$ rad/s. Visualize/compute the magnitude of the sum and argue why the filter passes low-frequency signals based on this graphical analysis.
   (c) Now do the same, but for a digital frequency corresponding to $\Omega = \pi/2$ rad/sample. Note this is far into the cut-off region, so we'd expect to see a fairly small magnitude response.
   (d) Plot the resulting magnitude and phase responses of the filters on the same axes, for ease of comparison. Feel free to use whatever matlab tools you have at your disposal. The magnitude response should be plotted as decibel gain $G = 20 \log_1 0|H|$ vs. $\log_{10} f$.
   (e) Compare and contrast the digital filter responses you see here to those you are accustomed to seeing in analog circuits. What's the same? What is different?

3. ***Real-world problems***: Describe a real-world signal processing problem you would like to pursue. Please provide enough details about the nature of the problem, the data sets, where digital filtering would come into play that your idea could become an Engn 395 reality. Just trying to get you think about a Choose Your Own Adventure problem!