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Frequency Sensitive Circuits: Phasor Math, Impedance, and RC filters

1 Math Stuff

Remember, there is a 3-way equivalence between 1) drawing a triangle; 2) writing a complex number $z = a + jb$, and writing a complex exponential $re^{j\phi}$. Be able to move between any one of these three forms

Adding complex numbers is straight forward, just remember to add the real parts separately from the imaginary parts

$$z_1 + z_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2). \tag{1}$$

Subtraction works similarly.

Exponentials are so nice to use because they have the following properties for multiplying and dividing complex numbers:

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)} \tag{2}$$

Note the amplitudes multiply; the angles add.

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_2-\theta_2)} \tag{3}$$

Note the amplitudes divide; the angles subtract.

Euler's ID is incredibly useful. We'll often use it in this form:

$$re^{j(\omega t+\phi)} = re^{j\phi}e^{j\omega t} \tag{4}$$

where r is the magnitude, $e^j\phi$ expresses the phase angle drawn in the complex plane, and ωt expresses the time dependence (how fast the phasor rotates around the plane in time). Notice the ***time dependence can be written separately from the magnitude and phase information.*** This is the mathemagic behind phasors. Also note that the a unit length complex exponential $e^{j\phi}$ simply encodes a rotation (phase angle) in the complex plane. When it multiplies another number, it does NOT change the magnitude.

Complex exponentials are ultimately just a very convenient mathematical tool for expressing sines and cosines. We need to know 3 things to define a wave: 1) amplitude; 2) phase; 3) (angular) frequency. Using Euler's ID we can write oscillating signals as follows

$$r\cos(\omega t + \phi) = Re\left[re^{j(\omega t+\phi)}\right] \tag{5}$$

1.1 Some math problems to get (re-)acquainted with complex numbers

Sketch each of the following in the complex plane. It may be helpful to write in $z = a + jb$ in complex exponential form $re^{j\phi}$. **Remember, r represents the magnitude and $e^{j\phi}$ represents the phase angle (rotation) relative to positive real axis!**

1. $5\,e^{j\pi/2}\,2\,e^{j\pi/4}$ $= 10 e^{j\,3\pi/4}$
2. $\frac{\sqrt{2}+j\sqrt{2}}{j}$ = rotate $e^{-j\pi/2}$
3. $\frac{3+j4}{e^{j\pi}}$ $= \frac{5 e^{j53^\circ}}{-1}$ START END
4. $10\,e^{j\pi/2}e^{j\pi/4}e^{j\pi/4}e^{-j\pi/4}e^{-j\pi/2}\,10\,e^{j\pi/2}$
   $100\, e^{j\,3\pi/4}$ $3\pi/4$

2 Capacitors

Capacitor action is based on the fact that capacitors store charge. For instance, equal and opposite charges can be stored on two conductive plates separated by an air gap (dielectric material). It takes energy to move charges around. This energy is related to the voltage of course. The fundamental relation for a capacitor is:

$$q_C(t) = Cv_C(t) \tag{6}$$

This relation says that voltage across the capacitor must be changing if the amount of charge is changing. Here C is the capacitance. It accounts for geometrical factors, such as the area of the conductors, the separation between them and the material properties of the dielectric. The larger the capacitor, the less the voltage (electrical) energy it takes to store charge on them. Capacitance is given in **units** of **Farads** [F].

Now let's find out what the ***impedance*** of a capacitor is!

1. Starting with Eqn 6, take a time derivative of both sides to show obtain the current $i_C(t)$ as a function of capacitance and voltage $v_C(t)$. Assume C is constant for now (actually many modern devices exploit a changing capacitance such that $dC/dt \neq 0$, but ignore that here).
2. Assume the voltage across the capacitor is given as $v_C(t) = v_o \cos(\omega t)$. Using Eqn 6, develop an expression for the inductor voltage $i_C(t)$.
3. Recall there are 3 attributes of any oscillating signal: magnitude, phase, and frequency. Which 1 of these is the *same* for both current and voltage in an capacitor? Which two are different?
4. Draw a phasor in the complex plane that represents the capactor current $i_C(t)$. We label this $\tilde{I}_C$. What is the magnitude of this phasor? What is the phase of this phasor (relative to the real axis)?
5. Then draw another phasor that represents the voltage $v_C(t)$. We label this $\tilde{V}_C$. What are its magnitude and phase?

6. Starting with the phasor $\tilde{I}_C$, what two math operations do you have to make in order to get it to have the same magnitude and phase as $\tilde{V}_C$? Hint: Look back at part c.

7. Now the punchline: We want to write a sort of generalized Ohm's Law for inductors:

$$\tilde{V}_C = \tilde{I}_C \tilde{Z}_C$$

The term $\tilde{Z}_C$ is what we call the electrical impedance of a capacitor. Again, you can think of this as a frequency dependent resistance. The impedance must therefore encode the math operations for scaling the magnitude and rotating the phase of $\tilde{I}$ to get to $\tilde{V}$. Therefore, show that:

$$\tilde{Z}_C = \underbrace{\frac{1}{\omega C}}_{scaling} \underbrace{e^{-\pi/2}}_{rot.} = \frac{-j}{\omega C} = \frac{1}{j\omega C}$$

8. Say you have a 0.56 μF capacitor in hand. What is the magnitude of its impedance $\left|\tilde{Z}_C\right|$ at a frequency of 10 Hz? At 100 Hz? At 1 kHz? Are these magnitudes "large" or "small" compared to say a 1.2 kΩ resistance? $\frac{1}{\omega C} = |Z_c| = \frac{1}{2\pi f C}$: $Z_c = \{28.4k\Omega, 2.8k\Omega, 280\Omega\}$

9. Assume you have a 1.2 kΩ resistor and a capacitor of 0.56 μF. At what angular frequency ω (rad/s) and corresponding frequency f (Hz) will these two components have the impedance **magnitude**, $|\tilde{Z}_C| = |\tilde{Z}_R|$? We'll see later this defines the ***cutoff frequency*** of a filter.
$R = \left|\frac{1}{j\omega C}\right| \Rightarrow \omega_0 = \frac{1}{RC} = 1488$ rad/s $\simeq 237$ Hz.

10. Let's think about some important limiting cases at extreme frequencies. What is the magnitude $\left|\tilde{Z}_C\right|$ at low frequencies (i.e., let $\omega \to 0$)? What is the magnitude at high frequencies (i.e., let $\omega \to \infty$)?

3 Resistors

Resistors you already know all about. They are easy mathematically: $v_R(t) = i_R(t)R$. Therefore, what is its impedance $\tilde{Z}_R$? You should be able to just write the answer down.

4 KVL, KCL, and Generalized Ohm's Law, Series and Parallel Equivalents

KVL and KCL are still true as ever. Energy must be conserved. Charge must be conserved. But now Ohm's Law has become a more general form of:

$$\tilde{V} = \tilde{I}\tilde{Z}.$$

Our solution strategies we employed for all dc circuits (e.g. voltage dividers with resistors only) will still be all the same, just now we are working with complex numbers to handle the frequency dependent ac circuits with impedance $\tilde{Z}$.

Note that series equivalent impedance work similarly as before:

$$\tilde{Z}_s = \sum_n \tilde{Z}_n = \tilde{Z}_1 + \tilde{Z}_2 + \ldots = \sum_n \tilde{Z}_n$$

Ditto for parallel equivalent impedance

$$\frac{1}{\tilde{Z}_p} = \sum_n \frac{1}{\tilde{Z}_n} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \ldots$$

5 High Pass Filter

At long last we finally get to see how to put all of this impedance and math stuff together to make a useful circuit. Figure 1 shows two classics. We'll start with the high pass filter

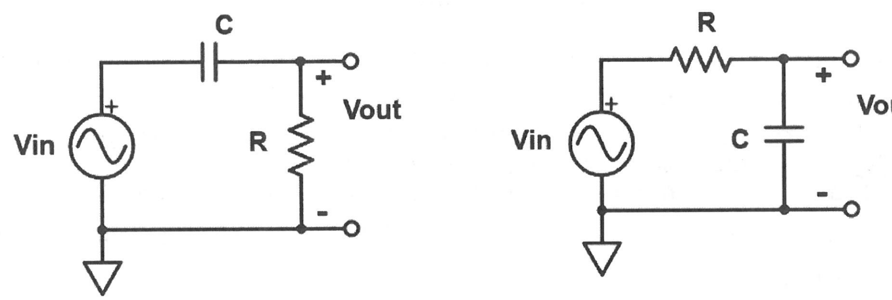

Figure 1: 1-stage (or 1-pole) RC filters. Left: high pass filter (HPF). Right: low pass filter filter (LPF)

1. Let's solve the high pass filter (HPF) first. Our approach will parallel what we did for voltage dividers with 2 resistors. After all, filters are just voltage dividers too! The just happen to be a frequency dependent voltage divider, in this case.

2. Where are we measuring the output (across which element)? Write a generalized Ohm's Law in the form of $\tilde{V}_{out} = \tilde{I}_x \tilde{Z}_x$. (where x is the subscript of the element we care about here). Don't plug in just yet for $\tilde{Z}_x$. $\tilde{V}_{out} = \tilde{V}_R = \tilde{I}_R \tilde{Z}_R$

3. Write KVL in the time domain: To help get you started, the gains are on the LHS. Write the drops on the RHS.

$$v_{in}(t) = \ldots \quad V_c(t) + V_R(t)$$

$e^{j\omega t}$ common term.

4. Assume sinusoidal response, where we have $v_{in}(t) = a_{in}\cos(\omega t + \phi_{in})$. Show for the HPF we can write KVL in phasor notation as:

$V_c(t) = a_c \cos(\omega t + \phi_c) = Re[\underbrace{a_c e^{j\phi_c}}_{\tilde{V}_c} e^{j\omega t}]$

$V_R(t) = a_R \cos(\omega t + \phi_R) = Re[\underbrace{a_R e^{j\phi_R}}_{\tilde{V}_R} e^{j\omega t}]$

$V_{in}(t) = Re[\underbrace{a_{in} e^{j\phi_{in}}}_{\tilde{V}_{in}} e^{j\omega t}]$

$$\tilde{V}_{in} = \tilde{V}_C + \tilde{V}_R$$

5. How does the resistor current and capacitor current compare? Write KCL using phasor notation.

i_c i_R $\quad i_c = i_R \Rightarrow \tilde{I}_c = \tilde{I}_R = \tilde{I}$

6. Using KCL,and Ohm's law, solve for the current $\tilde{I}$ in terms of $\tilde{V}_{in}$ and the impedances of $\tilde{Z}_R$ and $\tilde{Z}_C$.

7. Lastly, put all of the pieces of the puzzle together: solve for the ***transfer function*** in terms of the impedances:

$$\tilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{\tilde{Z}_R}{\tilde{Z}_C + \tilde{Z}_R} = \frac{R}{\frac{1}{j\omega C} + R} \cdot \underbrace{\left(\frac{j\omega C}{j\omega C}\right)}_{\text{fancy form of 1}}$$

Hint: this should look like a voltage divider relation, but with Z's instead of R's.

8. Plug in for the impedances to show that for the high pass filter we have:

$$\tilde{H}_{hpf}(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

Note the *hpf* subscript denotes this is the transfer function specific to a high pass filter. Other filters (and systems) will have different forms on the RHS. That said, the concept of transfer function = output/input is a general concept (throughout many branches of physics and engineering)

9. Magnitude relation: Show that the magnitude response is given by:

$$\left|\tilde{H}_{hpf}(\omega)\right| = \left|\frac{\tilde{V}_{out}}{\tilde{V}_{in}}\right| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

10. Make a decent (rough?) sketch of $\left|\tilde{H}_{hpf}(\omega)\right|$ vs. ω.

11. Cutoff frequency. At one special value of the frequency termed the cutoff frequency, ω_o, the magnitude relation above is $\left|\tilde{H}_{hpf}(\omega_o)\right| = 1/\sqrt{2}$. Compute the cutoff frequency in terms of R and C. Hint: Go back to step 8 and draw complex numbers representing the numerator and denominator.

12. Phase angle: Write a math expression of the for the phase angle of $\tilde{H}_{hpf}(\omega)$, which we call $\phi(\omega)$. This tells us how the phase of the output signal relative to the input signal. $\phi = \frac{\pi}{2} - \tan^{-1}(\omega RC)$

13. Last but not least, let's say we have a LPF with $R = 1.2k\Omega$ and $C = 0.56\,\mu F$. Compute the cutoff frequency for this HPF. Then imagine we have the following two waves input to the system. The low frequency reprsents the base line, the higher frequency the cow bell (always need more of that). Compute the corresponding output magnitude and phase for each. Which passes through? Which is attenuated?

   $\omega_o = 1488$ rad/s $= 237$ Hz

   (a) Input Wave 1: $v_{in}(t) = 0.5\cos(2\pi\ 20\,t)$ → $\frac{\omega}{\omega_o} = \frac{f}{f_o} = \frac{20}{237}$ $\quad |H| = 0.0841$, $\phi = 1.48$ rad $\approx \frac{\pi}{2}$

   (b) Input Wave 2: $v_{in}(t) = 0.5\cos(2\pi\ 2000\,t)$ → $\frac{f}{f_o} = \frac{2000}{237}$ $\quad |H| = 0.99 \approx 1$, $\phi = 0.118$ rad $= 6.7°$.

6 Low Pass Filter

Now let's solve for the LPF. The approach will parallel what we just did for the HPF in section 5. Trace back through the steps to solve for the LPF. Spoiler alert, you should arrive at at transfer function for the low pass filter of:

$\tilde{H}_{lpf}(\omega) = \frac{1}{1+j\omega RC}$