Assignment # 4: Stay Tuned!
LC oscillators and AM radio receiver (part I)
ENGN/PHYS 207—Fall 2019

Circuits You'll Build

- RLC oscillator
- AM radio receiver (part I)

Skills and Concepts You'll Learn

- Working with impedances $\tilde{Z}$ and transfer function $\tilde{H}(\omega)$.
- Using function generators to generate waves
- Using oscilloscopes measure ac signals
- Construction of capacitors and inductors

1 RLC resonators

Resonators are very important in circuits land. They are at the heart of wireless radio transmissions. Resonators are also at the heart of many chemical and biological engineering applications, such as sensing and differentiating certain chemicals (see Sanz et al).

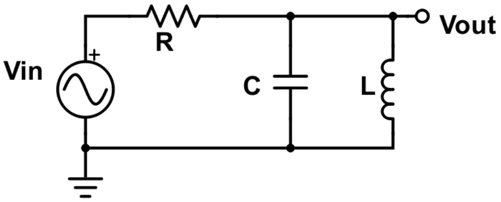


Figure 1: RLC resonator schematic. $R \approx 100\Omega$; $C \approx 100$ nF; $L = \approx$ 20-turn hand wound inductor. The output is measured across the LC parallel combo with respect to ground.

Fundamentally a resonator is circuit tuned to oscillate very strongly at one particular frequency, termed the ***resonance frequency***, set by the value of L and C. The classic resonance frequency is given by (units of Hz):

$$f_o = \frac{1}{2\pi\sqrt{LC}} \tag{1}$$

This is the frequency that the system naturally "wants" to oscillate at. A good mechanical analogy is a kid on a swing. The resonance frequency is set by the length of the swing (pendulum), and the gravitional constant. If the kid pumps their legs at just the right frequency—the resonance frequency—the kid will build up large magnitude oscillation swinging back and forth. Similarly, the RLC circuit will exhibit a large voltage output if the driving frequency (the "pumping of the legs") is timed just right. We'll start our journey into resonators by building, testing, analyzing the circuit shown in Figure 1.

2 Theoretical Considerations

First, let's get our feet wet working with resonators, as well as working with the transfer function $\tilde{H}(\omega) = \frac{\tilde{V_{out}}}{\tilde{V_{out}}}$. You must present theoretical work in the Appendix of your report.

1. We derived in class that the transfer function can be written as:

$$\tilde{H}(\omega) = \frac{\tilde{V_{out}}}{\tilde{V_{out}}} = \frac{\tilde{Z}_p}{\tilde{Z}_R + \tilde{Z}_p} \tag{2}$$

   where $\tilde{Z}_p$ is the parallel equivalent impedance of the inductor and capacitor.

   Show that the transfer function can be written as:

$$\tilde{H}(\omega) = \frac{j\omega L}{j\omega L + R(1 - \omega^2 LC)} \tag{3}$$

2. Show the ***magnitude*** of the transfer function is given by:

$$\left|\tilde{H}(\omega)\right| = \frac{\omega L}{\sqrt{(\omega L)^2 + \left(R(1 - \omega^2 LC)\right)^2}} \tag{4}$$

3. The angular frequency ω that maximizes the transfer function is termed the resonance frequency ω_o. Argue convincingly that the resonance frequency (rad/s) is given by:

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{5}$$

   and equivalently (in units of Hz)

$$f_o = \frac{1}{2\pi\sqrt{LC}} \tag{6}$$

4. Show the phase angle of the transfer function is given by:

$$\phi_H(\omega) = \frac{\pi}{2} - \tan^{-1}\left[\frac{\omega L}{R(1 - \omega^2 LC)}\right] \tag{7}$$

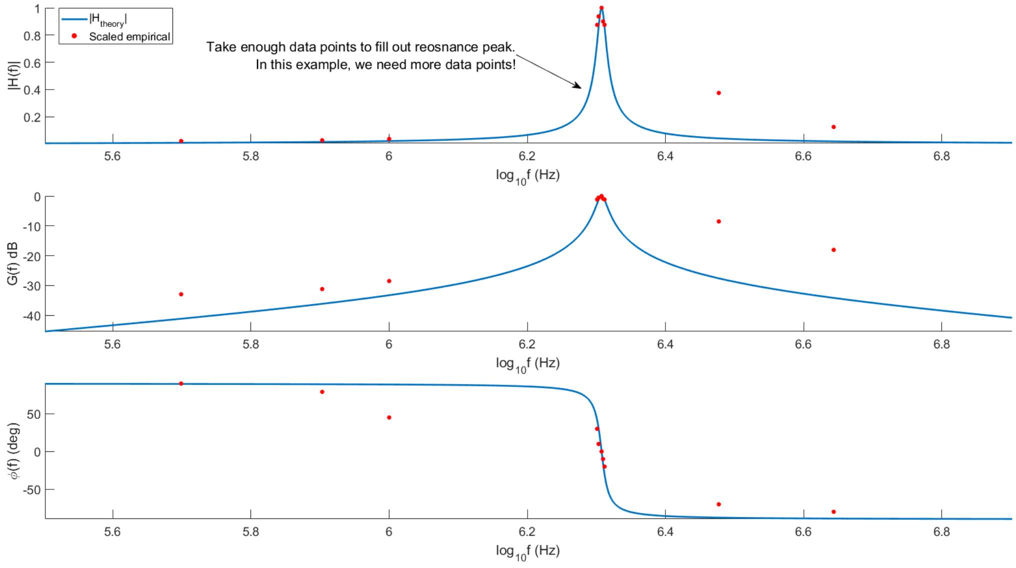
5. Show that the phase angle is zero at the resonance frequency: $\phi_H(\omega_o) = 0$.

At this point you are probably wondering "What on earth is the point of all this math-madness?!". Glad you asked, dear Circuits student! See the Math Appendix 5 for the answer. Read it carefully. It is critical for this lab and everything else you'll ever do in ac circuits that you understand this material!

3 Experiment and Analysis

Build the resonator circuit shown in Figure 1. The goal is to sweep out the transfer function magnitude $|\tilde{H}(f)|$and phase $\phi_H(f)$ as a function of frequency $f = \frac{\omega}{2\pi}$. Be sure to acquire sufficient data around the resonance peak. For example, see Figure 2—but be sure to acquire enough data points around the resonance peak to really define it will with empirical data. A couple of tips:

1. First thing you ought to do is quickly sweep through a wide range of frequencies from several kHz up to several MHz to approximately locate the resonance frequency.
2. Focus on gathering data points about your resonance peak first, moving in very small frequency increments. Plot the magnituide of the output wave vs. $\log_{10} f$ as you go to make sure you really swept out the resonance peak.
3. Farther from the resonance peak, choose frequencies with logarithmic spacing, e.g. work in powers of 4, for instance. So if your resonance peak was at 100 kHz, you can get a point at 100/4 kHz, 100/16 kHz, etc on the low side, and 400 kHz, 1600, kHz, etc on the upper side.
4. Phase measurements can be tricky. You can sometimes trust the scopes measured output, but *judge by your own eye whether the phase difference is positive or negative*!



Take enough data points to fill out reosnance peak.
In this example, we need more data points!

Figure 2: RLC resonance example experimental data (red dots) vs. theory (blue curves). The theory looks nice, hooray! But the empirical data is too sparse to really flesh out the resonance peak, boo!

The ultimate goal for this experiment is to characterize the resonator's behavior, and understand what it means. This is visualized with a series of a plots such as those shown in Figure 2 (but yours will be better!). There are 3 total plots:

1. $\left|\tilde{H}(f)\right|$ vs. $\log_{10} f$. Note: we can interchange angular frequency ω and f just so long as we keep very careful track of that pesky 2π.
2. Decibel gain $G(f)$ vs. $\log_{10} f$. (See math appendix)
3. Phase angle $\phi_H(f)$ vs. $\log_{10} f$.

Also note the logarithmic scale on the horizontal axis. This puts each order of magnitude on equal footing, making it easy to visualize over multiple ***decades*** (powers of 10) in frequency.

A couple of notes to help you in this little Circuits quest:

1. Make good use of the example Matlab code provided. Excel is soul-crushing at this point to type in such long formulas. Also, knowing Matlab actually will help land you spots in grad school and/or good jobs after graduation. I promise!
2. For visualization purposes, we will rescale the plot of magnitude vs. frequency data so that the resonance peak is shown as having a maximum of 1. In reality, you will likely not achieve real life measurements of $\left|\tilde{H}(\omega_o)\right| = 1$ owing to other sources of energy loss in the circuit (for instance, your oscilloscope leads form an inductor that siphons off some energy from your LC resonator). Again, this is just for ease of visual comparison in seeing a nice resonance peak.
3. Note: the phase data is plotted in degrees (and there is no need to rescale)

4 Tuning in: Guts of an AM radio receiver

4.1 Background on AM radio

Amplitude **M**odulated radio waves are still used round the world to broadcast sports, music, talk radio, emergency bulletins, etc. As we've previously seen in class, AM radio is transmitted on a carrier wave with a predefined frequency. For instance, in our neck of the woods, WREL 1450 AM broadcasts (aka transmits) information on a carrier wave with frequency $f_{carrier} = 1450$ kHz. The amplitude of this 1450 kHz wave is modulated according to the encoded audible waveform (Figure 3).

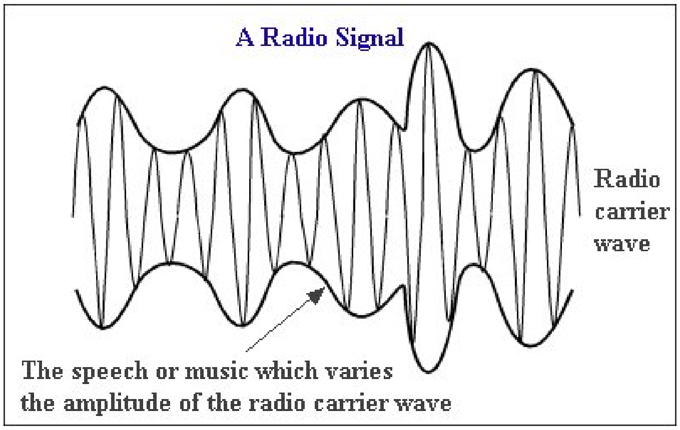


Figure 3: Principle of amplitude modulation. The carrier wave must oscillate many times within the sound envelope to make sure the envelope is well-defined. Image credit: `http://www.historywebsite.co.uk/Museum/Engineering/Electronics/history/valvedetails.htm`

The job of the AM radio receiver is to 1) *tune* into a particular station out of many available; and 2) *demodulate* the signal to recover only the audio signal. The first aim is accomplished with....drum roll, please...the ***LC resonator***.

4.2 Circuit Model and Theoretical Considerations

A little bit of theory to review. The AM radio LC resonator we are building today is modeled as pure inductor and pure capacitor in parallel. Energy is coupled into the parallel LC circuit via the inductor (Figure 4). This generates a current $i_{in}(t)$, which serves as the input into our circuit. The output is the voltage we measure across the parallel LC elements $v_{out}(t)$. So the output to input ratio—aka the transfer fucntion—is defined in this case as:

$$\tilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{I}_{in}}$$

Wait a second you say, a voltage divided by a current is a (drum roll, please...) impedance! So we can take one further step and write

$$\tilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{I}_{in}} = \tilde{Z}_{eq}$$

where $\tilde{Z}_{eq}$ is the equivalant parallel impedance of L and C. In summary, a wirelessly coupled radio signal generates a current which flows in the around the LC circuit.

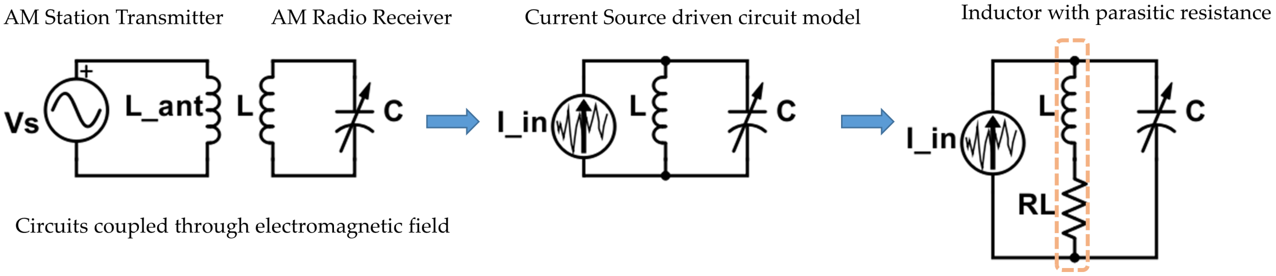


Figure 4: LC resonator for AM radio (left). Energy is coupled from the transmitter (base station) into the LC circuit via the inductor coupling (aka this acts like a ***transformer***). This system can be modeled as a current source pumping current (energy) into the resonator (middle). The inductor is more accurately modeled as having ***parasitic resistance*** R_L in series with the inductance L (right). The dotted peach-colored box indicates the actual physical inductor consisting of the series circuit elements.

We've seen that a resonator is able to tune into a certain frequency. Specifically, the maximal response occurs at the ***resonant frequency***:

$$\omega_o = \frac{1}{\sqrt{LC}} \text{ (rad/s) equivalently } f_o = \frac{1}{2\pi\sqrt{LC}} \text{ (Hz)}$$

As a friendly reminder, be careful with ω compared to f. They physically mean the same thing, its just that the angular frequency ω (rad/s) counts 2π rad for every oscillation, whereas f (Hz) simply counts how many times the wave oscillates per second The frequency response response is controlled by an parallel RC filter.

So we need an inductor and a capacitor for the tuning part of the circuit. Let's get building!

4.3 The Build

Today, you'll build your very own LC resonator, capable of being tuned to a desired AM radio station. It turns out Lexington, VA is almost in a radio free quiet zone, so the instructor will tell you next week about generating our own radio broadcast booth set to a carrier frequency in the range of 100-200 kHz. (WLUR guest DJ set, anyone?!). This is a bit lower than the typical AM frequency band which runs from about 560 kHz - 1.2 MHz for carrier frequencies. 100 kHz should still be plenty sufficient. Given the highest audio frequencies to transmit are about 5 kHz, this means we can fit in about $100/5 = 20$ peaks of the carrier to clearly define the audio envelope. Of course the more peaks within an audio oscillation the better.

The design and construction of the capacitor and inductor you'll build today was heavily inspired by this very nicely done video. (In general, beware circuits and electronics projects on the internets...there's a few good ones out there among an absolute pile of rubbish.)

4.4 Capacitor Construction

A capacitor is nothing more than two conductors separated by a dielectric (an insulating material). A capacitor can be made by two cylindrical shaped conductors. We'll use aluminum foil separated by a thin piece of paper. Yeah, MacGyver is smiling down. Check out the video link above between 30 s - about 2 min for construction details. Note that we also have copper (conductive) tape, in case that is helpful. Practical values for capacitor built in this way range from about 200 - 1000 pF = 0.2 - 1 nF. You can compute your expected capacitance as follows. Be sure to measure the range of achievable capacitances for your particular beautifully constructed one.

Cylindrical capacitor (in units of Farads (F)):

$$C = 2\pi\epsilon_o\epsilon_r\frac{L}{ln(b/a)}$$

where L = length of overlap between cylinders
b = outer radius
a = inner radius
ϵ_o = electric permittivity of free space (=8.85×10^{-12}) F/m
ϵ_r = relative permittivity (depends on the dielectric between the plates, value for paper is ≈ 3.85)

4.5 Inductor Construction

An inductor is nothing more than a loop of wire through which changing magnetic field flux can create an electromotive force aka a voltage. A classic construction technique for the inductor is to wrap enameled wire around a cylindrical object—a paper towel roll, PVC pipe, anything non-conductive. Note the very thin enamel acts as insulation, so you have to be sure to deinsulate the ends using sand paper, or a flame. Be sure to check you have connectivity before proceeding. How to do this? Well, inductors are just a big long wire, so you can measure the resistance. If you are properly connected, you should measure a value of on the order of 10 Ω. Inductance values for your own coil (in units of Henries (L)):

$$L_{coil} \approx N^2\mu_o\mu_r(D/2)\cdot\left[\ln\left(\frac{8D}{d}\right)-2\right]$$

where: N = number of turns
μ_o is the magnetic permeability of free space
μ_r is the relative magnetic permeability depending on what you've wrapped the coil around (=1 for air)
D = Diameter of the coil
d = the thickness of the wire (including insulation)

Even better, there's helpful calcultor here `https://www.eeweb.com/tools/coil-inductance` Be sure to select "coil" for the inductor type on the webpage. When choosing how the value for your inductor, hence, how many turns you need, remember that you want to set a resonance frequency in the range of 100-200 kHz.

4.6 Tuning in: LC resonator experiment

Now finagle any sensible wiring to get your homebrew capacitor and inductor in parallel. Copper tape can be helpful. So can aligator clips.

Next you need to create a homebrew broadcast station with the function generator and an antenna. Typically some sort of coil or long wire (a couple of meters should do) helps radiate the electromagnetic energy into space—well, just across your humble circuits lab abode. The function generator is essentially generating the carrier wave. You can sweep through a range of carrier frequencies by changing the frequency of the output waveform. Easy peasy!

The EM energy from your transmitter will couple into your homebrew LC resonant circuit. As a first test, you can move your antenna closer or farther from the LC tuning circuit. You should notice a big change in the LC resonantors voltage on the oscilloscope as you change the distance from the "broadcast booth."

1. Sweep out the transfer function $\tilde{H}(f)$ over a wide range of frequencies. Given you are trying for a resonant frequency of about 100 kHz, you should go a couple of decades above and a couple of decades below. Thus, 1 kHz - 10 MHz is a good range to test. Be sure to gather lots of data round the resonant peak!
2. Carefully plot your experimental data for $\tilde{H}(f)$, normalized such that the peak value is 1, vs. $\log f$. Use a software package to make a pretty plot of the normalized data. Use Matlab, as before! The course website has a juicy example waiting for you to download and use!
3. Use the experimental data plot to figure out the actual resonance frequency f_o of your LC circuit. Compare this to the theoretical $1/\sqrt{LC}$, given the L and C values computed and/or measured above. Also, given the measured resonance frequency and trusted and accurate measurement for the capacitance, what is a much better estimate of the actual inductance value L?
4. Also, use this plot to find the bandwidth of your resonator. The bandwidth is defined as:

   $$B = |f_H - f_L|$$

   The frequencies f_H and f_L are defined as the frequency where $\tilde{H}(f)$ falls to $1/\sqrt{2}$ of its maximum amplitude (This is -3 dB point in the decibel gain $G(f)$). The cutoffs should be easy to find because you should have already normalized the peak value of the experimental data plot to be 1. Also note that no data point likely falls exactly at $1/\sqrt{2}$, so it might be worthwhile to look at your data table to interpolate where the cutoff frequencies.
5. Compute the quality factor Q of your resonator. The Q value describes how selective the tuning is, or how "peaky" the transfer function is. A narrower bandwidth implies very high

selectivity, thus a higher Q, computed as:

$$Q = \frac{f_o}{B}.$$

6. Given your answers to the above, what must be the frequency spacing between multiple radio stations, such that there is no cross-talk between them. No cross-talk means that if you tune into 1450 AM, you don't hear competing sound bytes from say 1400 AM.

5 Appendix: Transfer Function Mathemagic

Basically, the transfer function takes an input signal, scales the magnitude and shifts its phase, then spits it out as the output signal. This ***scaling and phase shifting is frequency dependent***! Mathematically we formulate this as :

$$\tilde{V}_{out} = \tilde{H}(\omega)\tilde{V}_{in} \tag{8}$$

And remember that the phasors $\tilde{V}_{in}$ and $\tilde{V}_{out}$ encode the magnitude and phase of the input and output signals, respectively. They both oscillate with angular frequency ω.

Also remember that any complex number can be written in the form $z = re^{j\phi}$.

Thus, we can write Eqn 8 as follows:

$$v_{out}\, e^{j\phi_{out}} = r_H\, e^{j\phi_H}\, v_{in}\, e^{j\phi_{in}} = r_H v_{in}\, e^{j(\phi_H+\phi_{in})} \tag{9}$$

where $r_H = \left|\tilde{H}(\omega)\right|$, the magnitude of the transfer function.

So, we identify the output wave magnitude as $v_{out} = \left|\tilde{H}(\omega)\right| v_{in}$. This is just the input wave magnitude scaled by the magnitude of $\tilde{H}$.

Similarly, we identify the output wave phase angle as: $\phi_{out} = \phi_H + \phi_{in}$. This is just the input wave phase shifted by the phase of the transfer function.

And we know the output wave is oscillating at angular frequency ω, so we can write what the output wave!

$$V_{out}(t) = \left|\tilde{H}(\omega)\right| v_{in}\, \cos(\omega t + \phi_{in} + \phi_H)$$

. See how the magnitude

Also, one more thing: Clever folks over the years have adopted using the ***decibel gain*** $G(\omega)$ as an alternative representation to $\left|\tilde{H}(\omega)\right|$. The decibel [dB] gain is defined as:

$$G(\omega)\ [dB] = 20\log_{10}\left|\tilde{H}(\omega)\right| \tag{10}$$

The units are decibels (dB). Note that the decibel gain just puts the transfer function magnitude information on a logarithmic scale. It allows us to see more details in the regions far from the resonance frequency. We'll make use of the decibel gain over and over again!

6 What you need to turn in

6.1 RLC resonator

2 pages max, including figures. Appendix with theoretical work does not count toward page count.

1. Triplet of plots as shown in Figure 2.
2. Brief summary of findings comparing theory to experiment.
3. Theoretical work for “show that...” in section 2. This goes in an Appendix (in legible form!)

6.2 AM radio LC resonator

Update 17 Oct 2019: This week you won’t turn in anything in for the AM radio part. We’ll package that all up with next weeks’ lab into a single report on the full AM radio receiver.

1 page max, including figures. Appendix with theoretical work does not count toward page count.

1. Responses to all items in Section 4.6, including plots.
2. Brief summary of findings comparing theory to experiment.
3. All relevant theoretical work in appendix