ENGN/PHYS 207—Fall 2019
RC filters

1 Lowpass, Highpass, and Bandpass Filters

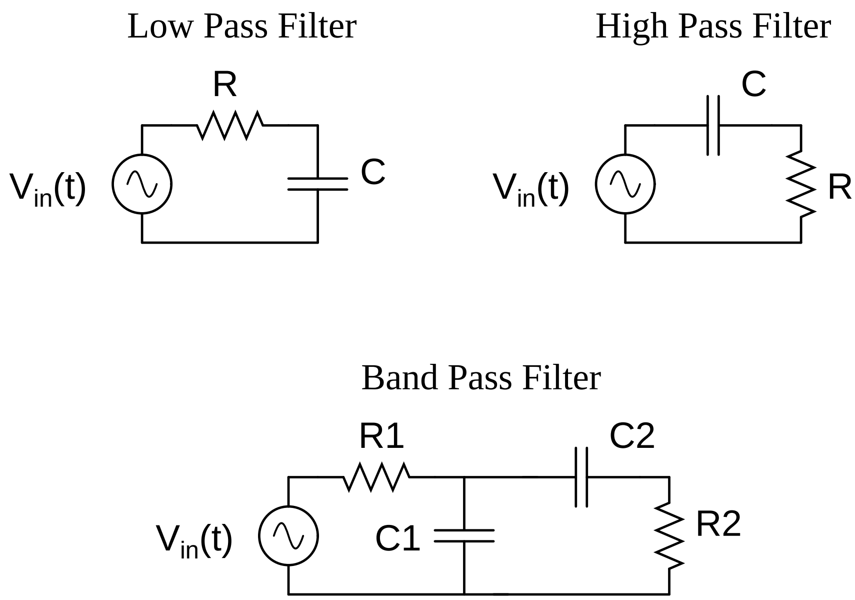

Figure 1: Passive RC filters. Output of LPF measured across C. Output of HPF measured across R. Output of BPF measured across R_2.

Figure 1 shows 3 types of filter that are very common:

1. 1-stage low pass filter (LPF)
2. 1-stage high pass filter (HPF)
3. Band pass filter (BPF) constructed form ***cascaded*** LPF + HPF.

The analysis we will do today is all based on inputting sinusoidal signals to the filters and evaluating the corresponding output signal. The fancy math term for this is *steady-state sinusoidal response*.

In terms of filter design, we always want to know the ***magnitude*** and ***phase*** response as a function of frequency. The transfer function $\tilde{H}_{LPF} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$ encodes both the magnitude response $|\tilde{H}(\omega)|$ and the phase response $\phi(\omega)$.

There are two numbers you will often hear that more or less characterize the filter:

1. cut-off frequency
2. slope in attenuation region (how "hard" the filter cuts off).

2 Low Pass Filter (LPF)

Figure 2 shows a 1st-order (aka "1-stage") low pass filter. "First order" because there is a single R and C.

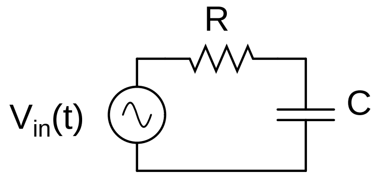

Figure 2: 1st order LPF. Output is measured across C.

1. Intuition can always be gained by examining limiting cases: $\omega = 0$ (low frequencies); and $\omega \to \infty$ (high frequencies). Draw the LPF for both cases. Hint: What does the capacitor act like at low frequencies? At high frequencies?

2. What is the output voltage you would expect to measure in these limiting cases? Therefore, does this filter pass low or high frequency components? "Pass" means the input signal appears basically unchanged at the output; the opposite is to "cut off" (or "attenuate").

3. Derive the transfer function for this filter:

$$\tilde{H}_{LPF}(\omega) = \frac{1}{1 + j\omega RC}$$

4. Carefully draw a phasor in the complex plane representing the numerator of $\tilde{H}_{LPF}(\omega)$. Draw another for the denominator.

5. For the numerator phasor, write an expression for the magnitude and phase angle. Do the same for the denominator.

6. There is a very special frequency termed the cutoff frequency, ω_o, for which the the magnitude response is: $\left|\tilde{H}_{LPF}(\omega_o)\right| = \frac{1}{\sqrt{2}}$. Given your answers to parts d and e above, show that the cutoff (angular) frequency is given by:

$$\omega_o = \frac{1}{RC} \qquad \text{or} \qquad f_o = \frac{1}{2\pi RC}.$$

   In a practical sense, we can choose appropriate R and C values in the lab to set the cutoff frequency! The resistor is usually chosen to be large enough to limit the current to something modest. As a rule of thumb, $R \geq 100\Omega$.

7. One might say "Gosh, that square root of 2 seems awfully arbitrary for a definition of the cutoff frequency!". Fair enough. However, it isn't really so. To gain a bit of intuition of why the cutoff is defined as such, show that the cutoff frequency is found by equation the impedance magnitude of resistor and capacitor. Hint: should be one line of math.

8. Show that the magnitude response as a function of frequency is:

$$\left|\tilde{H}_{LPF}(\omega)\right| = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

9. Now we'll consider some limiting cases. Show that for input signal frequencies way below the cutoff frequency ($\omega/\omega_o << 1$) that $\left|\tilde{H}_{LPF}(\omega)\right| \approx 1$. In other words, "signals with frequency less than the cutoff frequency pass through the filter."

10. Show that for input signal frequencies way above the cutoff frequency ($\omega/\omega_o >> 1$) that $\left|\tilde{H}_{LPF}(\omega)\right| \approx \frac{1}{\omega RC}$.

11. Next, time to play with the ***decibel gain***, defined as:

$$G(\omega) \ \text{[dB]} = 20\log_{10}\left|\tilde{H}(\omega)\right|$$

The units of $G(\omega)$ are "***decibels***" (dB), named after Alexander Graham Bell.

(a) What is the decibel gain for the LPF in the low frequency region, $\omega/\omega_o << 1$?

(b) Show that decibel gain for the LPF at the cutoff frequency is, $G(\omega_o) \approx -3$ dB.

(c) Show that a plot of $G(\omega)$ vs. $\log_{10}\omega$ has a slope of -20 dB/dec in the high frequency region $\omega/\omega_o >> 1$. Here "dec" refers to ***decade***, i.e., a factor of 10×.

(d) Given these 3 fun facts, make a sketch of the decibel gain over all frequencies.

12. Lastly, the phase response! Show that the phase response of the LPF is given by:

$$\phi_{LPF}(\omega) = 0 - \tan^{-1}(\omega RC) = -\tan^{-1}(\omega RC)$$

Sketch the phase as a function of (angular) frequency. What is the phase shift at the cutoff frequency?

3 High Pass Filter (HPF)

Figure 3 shows a 1st-order (aka "1-stage") high pass filter. "First order" because there is a single R and C.

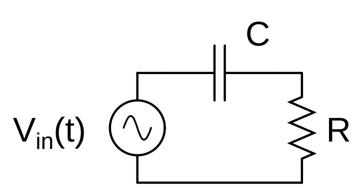

Figure 3: 1st order HPF. Output is measured across R.

1. Intuition can always be gained by examining limiting cases: $\omega = 0$ (low frequencies); and $\omega \to \infty$ (high frequencies). Draw the LPF for both cases. Hint: What does the capacitor act like at low frequencies? At high frequencies?

2. What is the output voltage you would expect to measure in these limiting cases? Therefore, does this filter pass low or high frequency components? "Pass" means the input signal appears basically unchanged at the output; the opposite is to "cut off" (or "attenuate").

3. Derive the transfer function for this filter:

$$\tilde{H}_{HPF}(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

4. Carefully draw a phasor in the complex plane representing the numerator of $\tilde{H}_{LPF}(\omega)$. Draw another for the denominator.

5. For the numerator phasor, write an expression for the magnitude and phase angle. Do the same for the denominator.

6. There is a very special frequency termed the cutoff frequency, ω_o, for which the the magnitude response is: $\left|\tilde{H}_{HPF}(\omega_o)\right| = \frac{1}{\sqrt{2}}$. Given your answers to parts d and e above, show that the cutoff (angular) frequency is given by:

$$\omega_o = \frac{1}{RC} \qquad \text{or} \qquad f_o = \frac{1}{2\pi RC}.$$

   In a practical sense, we can choose appropriate R and C values in the lab to set the cutoff frequency! The resistor is usually chosen to be large enough to limit the current to something modest. As a rule of thumb, $R \geq 100\ \Omega$.

7. Show that the magnitude response as a function of frequency is:

$$\left|\tilde{H}_{HPF}(\omega)\right| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

8. Now we'll consider some limiting cases. Show that for input signal frequencies way above the cutoff frequency ($\omega/\omega_o >> 1$) that $\left|\tilde{H}_{HPF}(\omega)\right| \approx 1$. In other words, "signals with frequency higher than the cutoff frequency pass through the filter."

9. Show that for input signal frequencies way below the cutoff frequency ($\omega/\omega_o << 1$) that $\left|\tilde{H}_{HPF}(\omega)\right| \approx \omega RC$.

10. Next, time to play with the ***decibel gain*** again.

    (a) What is the decibel gain for the HPF in the high frequency region, $\omega/\omega_o >> 1$?

    (b) Show that decibel gain for the LPF at the cutoff frequency is, $G(\omega_o) \approx -3$ dB.

    (c) Show that a plot of $G(\omega)$ vs. $\log_{10}\omega$ has a slope of +20 dB/dec in the low frequency region $\omega/\omega_o << 1$. Here, again, "dec" refers to ***decade***, i.e., a factor of 10×.

    (d) Given these 3 fun facts, make a sketch of the decibel gain over all frequencies.

11. Lastly, the phase response! Show that the phase response of the HPF is given by:

$$\phi_{HPF}(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega RC)$$

Sketch the phase as a function of (angular) frequency. What is the phase shift at the cutoff frequency?

4 Band Pass Filter (BPF)

We can make a band pass filter by appropriately cascading a high-pass and low-pass filter. A band pass filter essentially sets both an lower and upper limit of input signal frequency components that are allowed to pass through the BPF. Frequencies that "pass" are between lower and upper frequency limits, ω_L and ω_H (rad/s), respectively. Similarly, we can denote the low and high cutoffs of the BPF f_L and f_H (Hz).

Figure 4 shows the simplest possible BPF. The output is measured across R_2. One very important counter intuitive consideration: Note that the low-pass section (R_1C_1) sets the *upper* cutoff limit ω_H (or f_H). Similarly, the high-pass section (R_2C_2) sets the *lower* cutoff limit ω_L (or f_L). Classic mistake in the lab is to think of "high pass filter –¿ upper limit" and "low pass filter –¿ lower limit". Don't fall into this trap, or your filter function will be pretty sad :(.

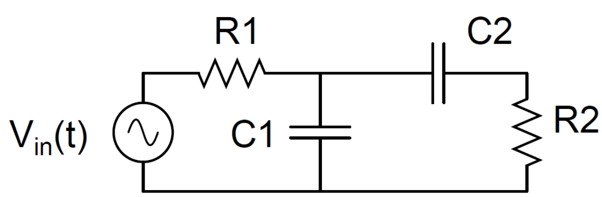

Figure 4: Band pass filter constructed by cascading low pass and high pass elements. Output is measured across R_2.

1. To get intuition about the band pass, draw the limiting case circuits. By now you know the drill, consider both $\omega = 0$ and $\omega \to \infty$.

2. Given the schematics for the limiting cases, argue that the output voltage is in fact zero for both very low and very high frequencies.

3. If the LPF and HPF sections are properly cascaded (see important considerations below!), we can think of the transfer function as follows:

$$\tilde{H}_{BPF}(\omega) = \tilde{H}_{LPF}(\omega)\,\tilde{H}_{HPF}(\omega).$$

Thus, show we can write:

$$\tilde{H}_{BPF}(\omega) = \left(\frac{1}{1 + j\omega R_1 C_1}\right)\left(\frac{j\omega R_2 C_2}{1 + j\omega R_2 C_2}\right)$$

4. Furthermore show the lower and upper band pass filter cutoff frequencies are given by: $f_L = f_{hpf} = \frac{1}{2\pi R_2 C_2}$; $f_H = f_{lpf} = \frac{1}{2\pi R_1 C_1}$
5. Use your results above for the LPF and HPF to make a quick sketch of the decibel gain over all frequencies. Hint:
$$20\log_{10} AB = 20\log_{10} A + 20\log_{10} B.$$
6. User your results for the LPF and HPF above to make a quick sketch of the phase response over all frequencies. Also, show that the phase response is given by:
$$\phi_{bpf}(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega R_1 C_1) - \tan^{-1}(\omega R_2 C_2)$$

5 Cascading RC filters: The 10x rule

Higher order passive RC filters can be constructed from single stages. For instance, the BPF above (Figure 4 cascades a low pass and high pass section together.

In order for each section further "downstream" (later in the circuit) to not affect what was happening "upstream" (earlier in the circuit), we must adhere to a 10× rule of thumb: Proper cascades of passive filters require each successive stage to use a resistor (at least) a factor of 10× greater than the one in the previous stage. This prevents current from leaking out of the earlier stages, which would torpedo its operation.

For instance, if in the BPF we use $R_1 = 1$ kΩ, then R_2 must be at least 10 kΩ. If we were to add a third stage, we would require $R_3 \geq 100$ kΩ.

In terms of cascades, not we can make other fun filters such as a 2-stage HPF by cascading two individual HPF sections together, typically chosen to have $R_1C_1 = R_2C_2$. The 2-stage cuts off twice as hard as the 1-stage (40 dB/dec slope). A three stage would cutoff at 60 dB/dec.

Similarly, we could make a 2- or 3-stage LPF or HPF. For example, see figures 6 and 7.

As another example, we could make a BPF that has 3 total sections: 1 LPF cascaded with 2 HPF sections. The single LPF section "softly" cuts off the high high frequencies while the 2 HPF sections cascaded more strongly cutoff the low frequencies. This is shown in Figure 5.

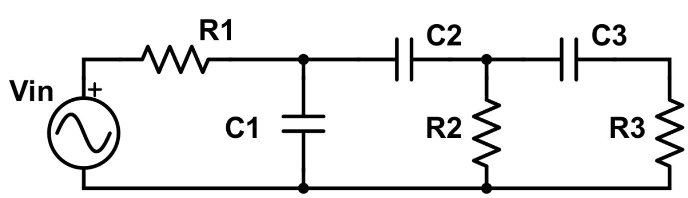

Figure 5: BPF cascasde. In this example, we have 1 LPF section with 2 HPF sections. Proper construction requires that $R_1 << R_2 << R_3$.

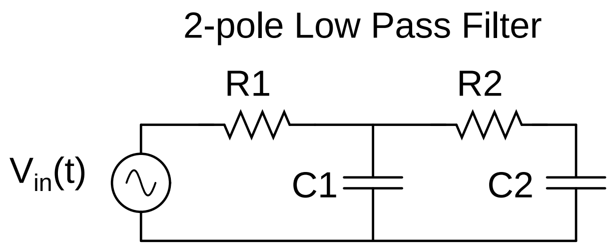

Figure 6: 2nd order LPF. Proper construction requires that $R_1 << R_2$, and typically we choose $R_1C_1 = R_2C_2$.

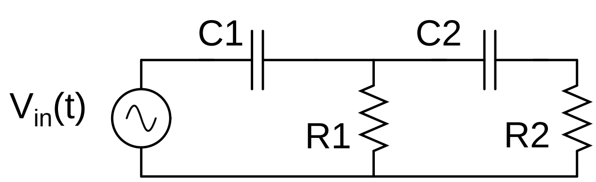

Figure 7: 2nd order HPF. Proper construction requires that $R_1 << R_2$, and typically we choose $R_1C_1 = R_2C_2$.